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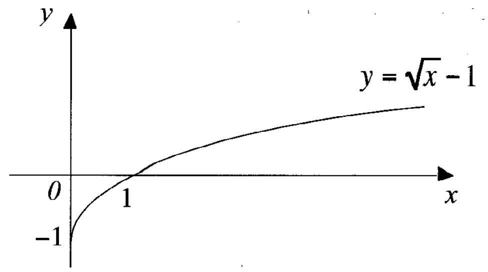
Question 1 **Begin a new page**

CSSA 2006 Ext.2 trial

(a)



The diagram shows the graph of the function $f(x) = \sqrt{x} - 1$. Use the graph of $y = f(x)$ to sketch (on separate diagrams) the following graphs, showing the values of any intercepts on the coordinate axes and the equations of any asymptotes:

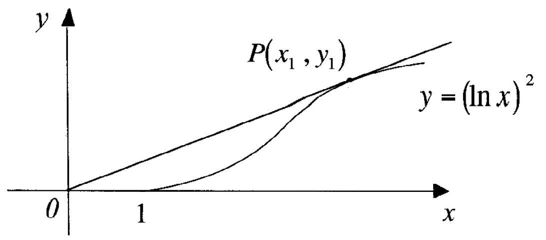
(i) $y = |f(x)|$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = \tan^{-1} f(x)$ **2**

(b)



The diagram shows the graph of the function $f(x) = (\ln x)^2$, $x \geq 1$. $P(x_1, y_1)$ is a point on the curve such that the tangent to the curve at P passes through the origin O.

(i) By considering the gradient of the line OP in two different ways, show that P is the point $(e^2, 4)$. **2**

(ii) Find the set of values of the real number k such that the equation $f(x) = kx$ has two distinct real roots. **1**

(iii) Use integration by parts to show that $\int (\ln x)^2 \, dx = x(\ln x)^2 - 2x\ln x + 2x + c$. **4**

Hence find the exact area of the region bounded by the curve $y = f(x)$, the x axis and the line OP.

(iv) Find the equation of the inverse function $f^{-1}(x)$. **1**

(v) Find the equation of the tangent to the curve $y = f^{-1}(x)$ that passes through the origin. **1**

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Question 2 **Begin a new page**

(a) Evaluate $\int_0^4 \frac{1}{\sqrt{x^2+9}}\,dx$, giving the answer in simplest exact form. **2**

(b) Evaluate $\int_0^1 e^x \cos\left(e^x\right)\,dx$, giving the answer correct to 4 significant figures. **2**

(c) Evaluate $\int_0^2 \frac{x(x-16)}{(4x+1)(x^2+4)}\,dx$, giving the answer in simplest exact form. **4**

(d) Use the substitution $u = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3\cos x - 4\sin x + 5}\,dx$. **4**

(e) $f(x)$ is a continuous, odd function. Use the substitution $u = -x$ to show that $\int_{-a}^{a} f(x)\,dx = 0$. **3**

Question 3 **Begin a new page**

(a) Find the values of real numbers a and b such that $\frac{a}{i} + \frac{b}{1+i} = 1$. **3**

(b)(i) Express $z = 1 + i$ in modulus / argument form. Hence show that $z^9 = 16z$. **2**

(ii) Express $(1+i)^9 + (1-i)^9$ in the form $a + ib$ where a and b are real. **2**

(c) In the Argand diagram points $A,\ B,\ C,\ D$ represent the complex numbers $\alpha,\ \beta,\ \gamma,\ \delta$ respectively.

(i) If $\alpha + \gamma = \beta + \delta$ show that $ABCD$ is a parallelogram. **2**

(ii) If $ABCD$ is a square with vertices in anticlockwise order, show that $\gamma + i\alpha = \beta + i\beta$. **2**

(d)(i) In the Argand diagram shade the region where both $\left|z - (1+i)\right| \le 1$ and $0 \le \arg\left(z - (1+i)\right) \le \frac{\pi}{4}$. **2**

(ii) Find the sets of values of $|z|$ and of $\arg z$ for points in the shaded region. **2**

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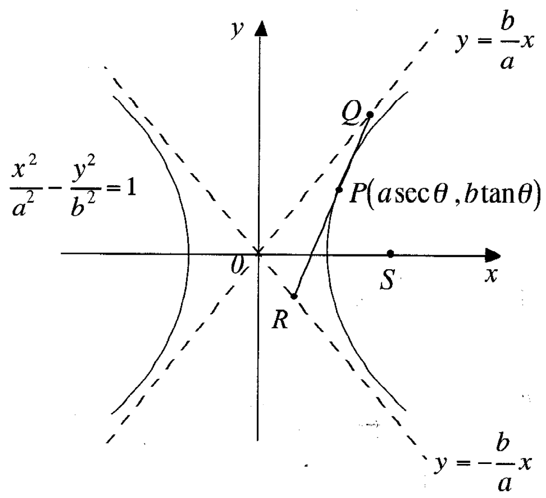
Question 4 **Begin a new page**

(a)



In the diagram $P(a\sec\theta\,, b\tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. S is a focus of the hyperbola. The tangent to the hyperbola at P meets the asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ at the points Q and R respectively.

(i) Show that the tangent to the hyperbola at P has equation $bx\sec\theta - ay\tan\theta = ab$. **2**

(ii) Show that Q and R have coordinates $\left(a(\sec\theta + \tan\theta)\,, b(\sec\theta + \tan\theta)\right)$ and $\left(a(\sec\theta - \tan\theta)\,, -b(\sec\theta - \tan\theta)\right)$ respectively. **2**

(iii) Show that P is the midpoint of QR. **1**

(iv) Show that $OQ \times OR = OS^2$ where O is the origin. **3**

(b) $P\left(p\,, \frac{1}{p}\right)$ and $Q\left(q\,, \frac{1}{q}\right)$ are two points on the rectangular hyperbola $xy = 1$. M is the midpoint of the chord PQ.

(i) Show that the chord PQ has equation $x + pqy - (p+q) = 0$. **2**

(ii) If P and Q move on the rectangular hyperbola such that the perpendicular distance of the chord PQ from the origin $O(0\,,0)$ is always $\sqrt{2}$, show that $(p+q)^2 = 2(1 + p^2q^2)$. **1**

(iii) Hence find the equation of the locus of M, stating any restrictions on its domain and range. **4**

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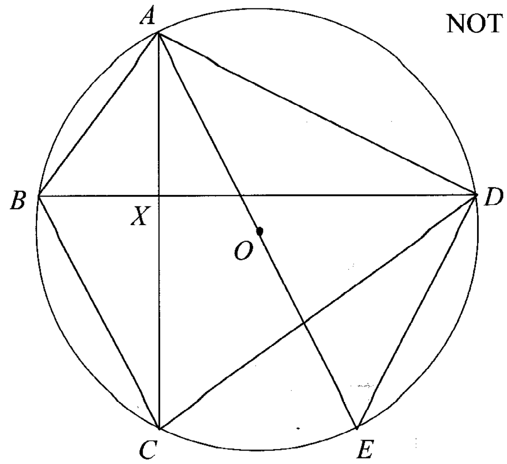
Question 5 **Begin a new page**

(a)



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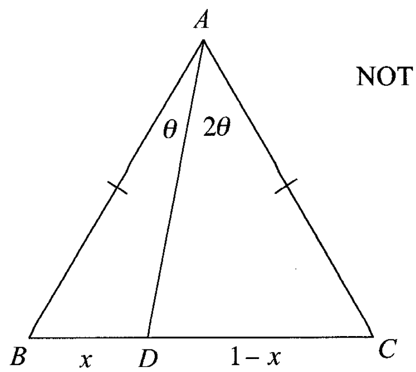
In the diagram, AE is a diameter of a circle with centre O. Quadrilateral $ABCD$ is inscribed in the circle. The diagonals AC and BD intersect at right angles at the point X.

(i) Copy the diagram

(ii) Show that $\Delta ABX \,|||\, \Delta AED$ and deduce that $BC = ED$. **4**

(iii) Hence show that $AX^2 + BX^2 + CX^2 + DX^2 = d^2$, where d is the diameter of the circle. **4**

(b)



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In the diagram ABC is a triangle in which $AB = AC$ and $BC = 1$. D is the point on BC such that $\angle BAD = \theta$, $\angle CAD = 2\theta$, $BD = x$ and $CD = 1 - x$.

(i) Use the sine rule in each of ΔADB and ΔADC to show that $\cos\theta = \dfrac{1-x}{2x}$. **4**

(ii) Hence show that $\frac{1}{3} < x < \frac{1}{2}$. **3**

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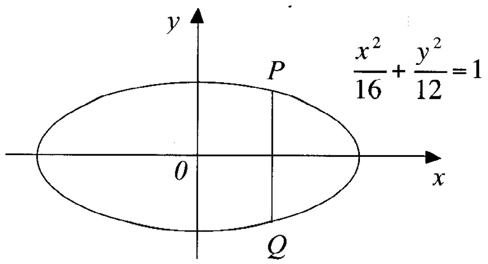
Question 6 **Begin a new page**

(a) T_n , $n = 1, 2, 3, \ldots$ is a sequence of positive integers. S_n , $n = 1, 2, 3, \ldots$ is another sequence of positive integers such that $S_n = T_1 + T_2 + T_3 + \ldots + T_n$. Also $S_1 = 6$, $S_2 = 20$ and $S_n = 6\,S_{n-1} - 8\,S_{n-2}$, $n = 3, 4, 5, \ldots$.

(i) Use Mathematical Induction to show that $S_n = 4^n + 2^n$, $n = 1, 2, 3, \ldots$. **5**

(ii) Hence find T_n , $n = 1, 2, 3, \ldots$ in simplest form. **3**

(b)



In the diagram the line $x = 2$ meets the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$ at the points P and Q. A solid has as its base the region $\left\{(x\,,\,y)\ :\ \dfrac{x^2}{16} + \dfrac{y^2}{12} \le 1 \text{ and } x \ge 2\right\}$. Each cross section perpendicular to the y axis is a square with one side in the base of the solid.

(i) Show that the volume V of the solid is given by $V = \displaystyle\int_{-3}^{3} (x-2)^2 \; dy$. **3**

(ii) Hence find the value of V in simplest exact form. **4**

Marks

Question 7 **Begin a new page**

(a) A body of mass m kg is moving in a horizontal straight line. At time t seconds it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} and acceleration a ms^{-2}. The body is subject to a resistance of magnitude $\frac{1}{10}m\sqrt{v}\left(1+\sqrt{v}\right)$ Newtons. Initially the body is at O and has velocity V ms^{-1}.

(i) Show that $a = -\frac{1}{10}\sqrt{v}\left(1+\sqrt{v}\right)$. **1**

(ii) Show that $t = -10\int \frac{1}{\sqrt{v}\left(1+\sqrt{v}\right)}\,dv$. Hence find an expression for t in terms of v. **3**

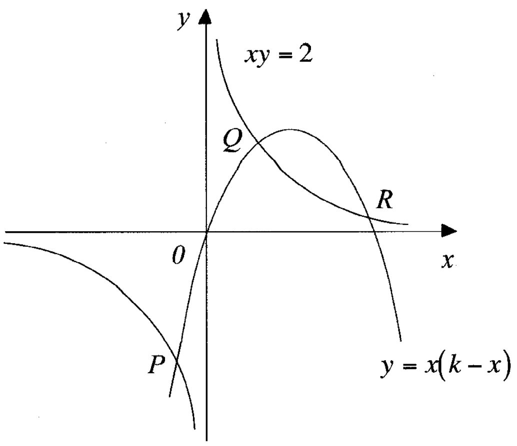
(You may use the substitution $v = u^2$ if required.)

(iii) Show that $x = -10\int \frac{\sqrt{v}}{1+\sqrt{v}}\,dv$. Hence find an expression for x in terms of v. **3**

(You may use the substitution $v = u^2$ if required.)

(iv) Find the distance travelled and the time taken in coming to rest. **2**

(b)



In the diagram the curves $xy = 2$ and $y = x\left(k-x\right)$ intersect at the points P, Q and R with x coordinates α, β and γ respectively.

(i) Show that α, β and γ satisfy the equation $x^3 - kx^2 + 2 = 0$. **1**

(ii) Find the value of k such that α, β and γ are consecutive terms in an arithmetic sequence. **3**

(iii) Find the monic cubic equation with coefficients in terms of k whose roots are α^2, β^2 and γ^2. **2**

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Question 8 **Begin a new page**

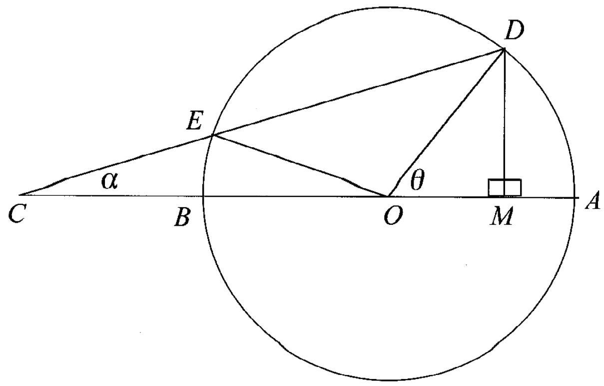
(a)(i) Solve the equation $z^5 - 1 = 0$, giving the roots in modulus / argument form. **2**

(ii) Hence show that $z^5 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$. **2**

(iii) Show that $4\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = 5$. **2**

(iv) Hence show that $x = \cos\frac{2\pi}{5}$ is a root of the equation $8x^3 - 8x^2 - 8x + 3 = 0$. **2**

(b)



NOT TO SCALE

In the diagram AB is a diameter of a circle with centre O and radius 1. C is a point on AB produced such that $BC = AO = OB$. D is a point on the circle such that $\angle AOD = \theta$, $0 < \theta < \frac{\pi}{2}$. CD cuts the circle at E and $\angle BCE = \alpha$. M is the foot of the perpendicular from D to AB.

(i) Show that $\tan\alpha = \dfrac{\sin\theta}{2 + \cos\theta}$. **2**

(ii) Explain why $\angle BOE = \alpha + \varepsilon$ for some $\varepsilon > 0$. Hence show that $\theta = 3\alpha + \varepsilon$. **3**

(iii) Hence show that $\dfrac{\sin\theta}{2 + \cos\theta} < \tan\dfrac{\theta}{3}$, $0 < \theta < \frac{\pi}{2}$. **2**