Ext 2 Trial Solutions 2011

Q 1

a) $$\int \frac{1+x}{4+x^2}\,dx = \int \frac{1}{4+x^2}\,dx + \frac{1}{2}\int \frac{2x}{4+x^2}\,dx$$
$$= \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln\left(4+x^2\right) + c$$

b) $$\int \frac{1}{\sqrt{6-x^2-x}} = \int \frac{1}{\sqrt{-(x^2+x-6)}}$$
$$= \int \frac{1}{\sqrt{-\left[\left(x^2+x+\frac{1}{4}\right)-\frac{25}{4}\right]}}$$
$$= \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}}$$
$$= \sin^{-1}\frac{x+\frac{1}{2}}{\frac{5}{2}} + c$$
$$= \sin^{-1}\frac{2x+1}{5} + c$$

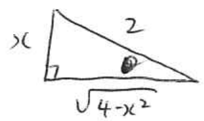
c) $$\int \sin^3 x\cos^3 x\,dx = \int \sin^3 x\cos x\left(1-\sin^2 x\right)dx$$
$$= \int \sin^3 x\cos x - \int \sin^5 x\cos x\,dx$$
$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c$$

d) $$\int_0^{\ln 2} x e^{-x}\,dx$$ let $u = x$, $du = dx$; $dv = e^{-x}dx$, $v = -e^{-x}$
$$= \left[-xe^{-x}\right]_0^{\ln 2} + \int_0^{\ln 2} e^{-x}\,dx$$
$$= -\ln 2\, e^{-\ln 2} - 0 + \left[-e^{-x}\right]_0^{\ln 2}$$
$$= -\ln 2 \times \tfrac{1}{2} + \left(-e^{-\ln 2} - -e^{0}\right)$$
$$= \frac{-\ln 2}{2} - \frac{1}{2} + 1 = \frac{1-\ln 2}{2}$$

Q 1 e) $$\int \frac{\sqrt{5-x}}{\sqrt{5+x}}\cdot\frac{\sqrt{5-x}}{\sqrt{5-x}} = \int \frac{5-x}{\sqrt{25-x^2}}\,dx$$
$$= \int \frac{5}{\sqrt{25-x^2}}\,dx + \frac{1}{2}\int \frac{-2x}{\sqrt{25-x^2}}\,dx$$
$$= 5\sin^{-1}\frac{x}{5} + \sqrt{25-x^2} + c$$

f) $$I = \int \frac{dx}{x^2\sqrt{4-x^2}}$$

Construct right angled Δ with smaller length $\sqrt{4-x^2}$:



$\Rightarrow \sin\theta = \frac{x}{2}$

$x = 2\sin\theta$

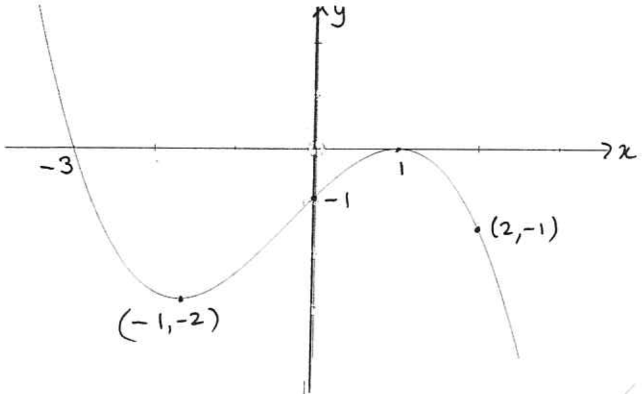
$dx = 2\cos\theta\,d\theta$

also, $\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$\therefore \sqrt{4-x^2} = 2\cos\theta$

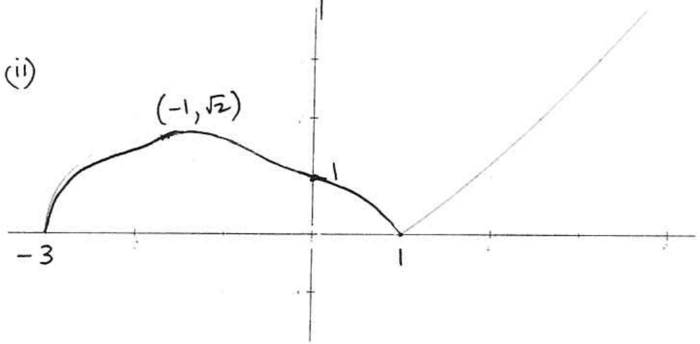
$$\therefore I = \int \frac{2\cos\theta\,d\theta}{4\sin^2\theta\cdot 2\cos\theta}$$
$$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$
$$= \frac{1}{4}\int \operatorname{cosec}^2\theta\,d\theta$$
$$= -\frac{1}{4}\cot\theta + c$$
$$= -\frac{\sqrt{4-x^2}}{4x} + c$$

2) (a) (i) All diagrams are not strictly to scale. ③



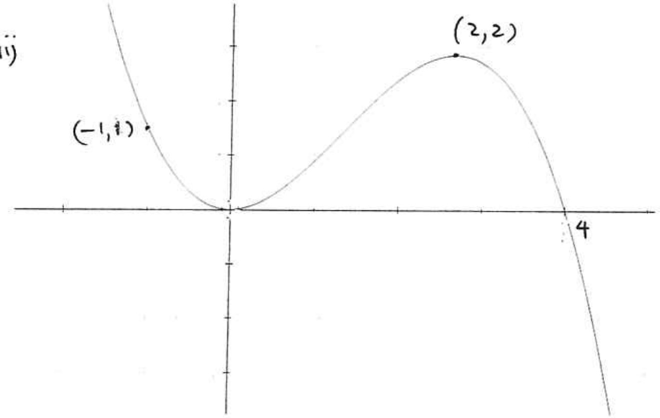
$y = -f(x)$

(ii)



$y = \sqrt{f(x)}$

(iii)



$y = f(1-x) = f(-(x-1))$

flip over y axis and shift 1 to the right

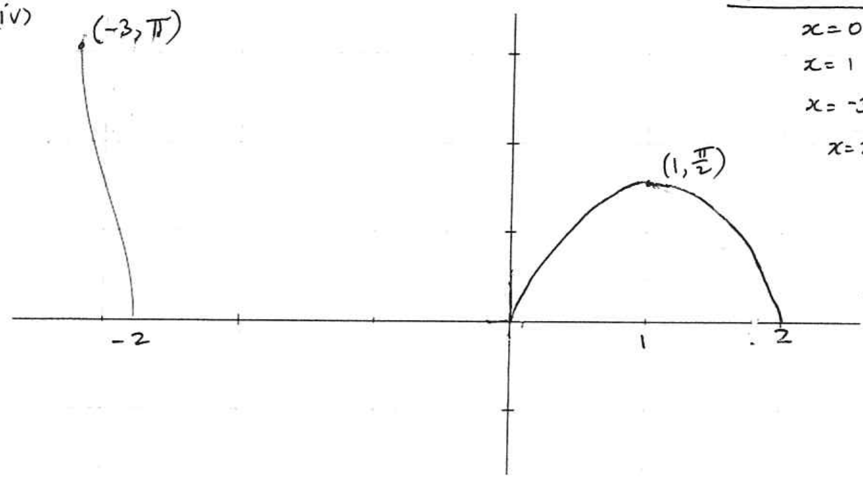
$x=0 \quad f(1)=0$
$x=1 \quad f(0)=1$
$x=2 \quad f(-1)=2$
$x=-1 \quad f(2)=1$
$x=3 \quad f(-2)=1$
$x=4 \quad f(-3)=0$

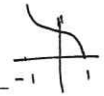
④

$y = \cos^{-1}(f(x))$

$x=0 \quad \cos^{-1}1=0$
$x=1 \quad \cos^{-1}0=\frac{\pi}{2}$
$x=-3 \quad \cos^{-1}-1=\pi$
$x=2 \quad \cos^{-1}1=0$

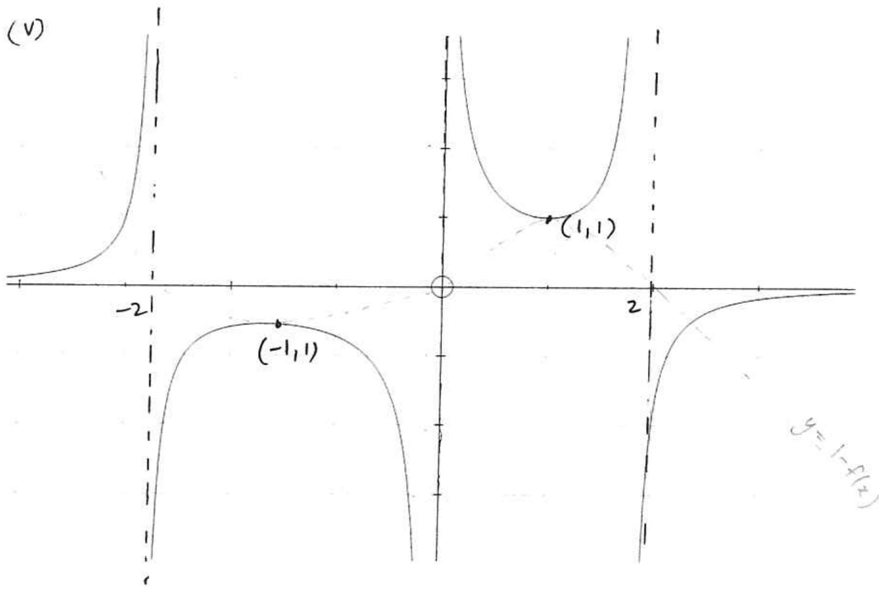
(iv)





Gap between $x=-2$ and $x=0$ as $f(x)>1$ [not defined for $\cos^{-1}$

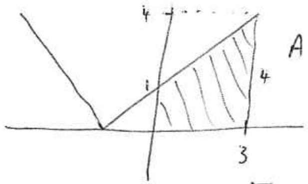
(v)



Sketch $y = 1-f(x)$ (shift (i) up by 1 unit)
Then sketch reciprocal.

Q2 b) $P(x) = (x^2-3)(x-4)$

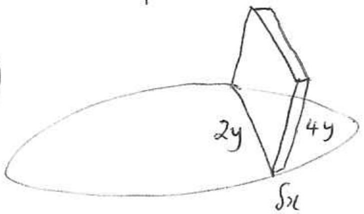
c) $\int_0^3 |x+1|\,dx$ = area under graph:



$A = \frac{1}{2}\cdot 3\,(1+4)$

$= \frac{15}{2}$

(d)



typical rectangular prism has vol:

$= 8y^2\,\delta x$

$\therefore V = \int_{-4}^{4} 8y^2\,dx$

given the symmetry $V = 2\int_0^4 8y^2\,dx$

$= 16\int_0^4 16 - x^2\,dx$

$= 16\left[16x - \frac{x^3}{3}\right]_0^4$

$= 16\left[\left(64 - \frac{64}{3}\right) - 0\right]$

$= \frac{2048}{3}\,u^3$

Q3

(a)(i) $\frac{2-3i}{1+i}\cdot\frac{1-i}{1-i}$

$= \frac{2-2i-3i+3i^2}{1+i}$

$= -\frac{1}{2} - \frac{5i}{2}$

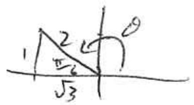
$\therefore \overline{z} = -\frac{1}{2} + \frac{5i}{2}$

(ii) $|z| = \sqrt{\frac{1}{4}+\frac{25}{4}}$

$= \frac{\sqrt{26}}{2}$

(b)(i) $\omega = -\sqrt{3} + i$

$\therefore \omega = 2\operatorname{cis}\frac{5\pi}{6}$



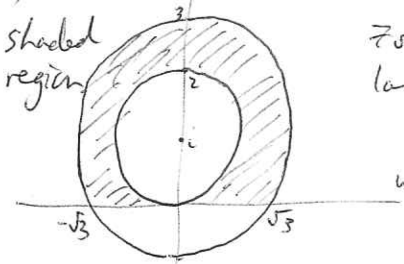
(ii) $\omega^7 + 64\omega = 2^7\operatorname{cis}\frac{35\pi}{6} + 64\cdot 2\operatorname{cis}\frac{5\pi}{6}$

$= 128\operatorname{cis}\left(-\frac{\pi}{6}\right) + 128\operatorname{cis}\frac{5\pi}{6}$

$= -128\operatorname{cis}\frac{5\pi}{6} + 128\operatorname{cis}\frac{5\pi}{6}$

$= 0$

(c)



For x intercepts, larger circle given by

$x^2 + (y-1)^2 = 4$

when $y = 0$

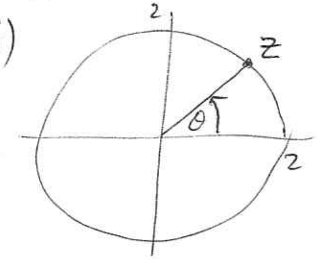
$x^2 + 1 = 4$

$x^2 = 3$

$x = \pm\sqrt{3}$

Q3 (d)

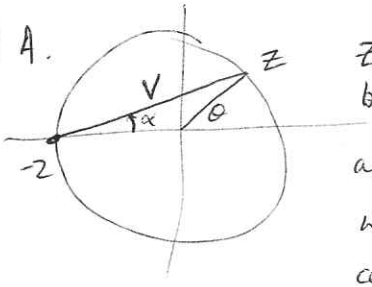
(i)



(ii) $\arg z^2 = 2\arg z$

$= 2\theta$

(iii) A.

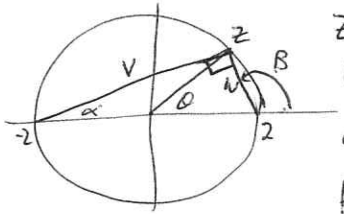


$z+2$ can be represented by the vector, V.

$\arg(z+2) = \alpha$

where $\alpha = \frac{\theta}{2}$ (angle at the centre = twice angle at the circumference)

B



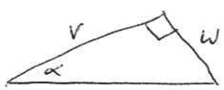
$z-2$ can be represented by the vector, W

$\arg(z-2) = \beta$

$\beta = \alpha + 90°$ (ext angle of Δ, the Δ is right-angled (angle in semi-circle))

$= \frac{\theta}{2} + 90$

C. $\left|\frac{z-2}{z+2}\right| = \frac{|z-2|}{|z+2|} = \frac{|w|}{|v|}$



Now $\tan\alpha = \frac{|w|}{|v|}$

i.e $\tan\frac{\theta}{2} = \frac{|w|}{|v|}$

Q4

a) $xy = c^2$

(i) $M_{PQ} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{c\,\frac{q-p}{pq}}{c(p-q)} = \frac{-\frac{(p-q)}{pq}}{(p-q)} = -\frac{1}{pq}$

$\therefore y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$

$pqy - cq = -x + cp$

$x + pqy = c(p+q)$

(ii) Chord at x axis : $x = c(p+q)$ i.e $A[(c(p+q), 0)]$

at y axis : $pqy = c(p+q)$ i.e $B\left(0, \frac{c(p+q)}{pq}\right)$

$(d_{AP})^2 = (cp - c(p+q))^2 + \left(\frac{c}{p}\right)^2$

$= c^2q^2 + \frac{c^2}{p^2}$

$(d_{BQ})^2 = (cq - 0)^2 + \left[\frac{c}{q} - \frac{c(p+q)}{pq}\right]^2$

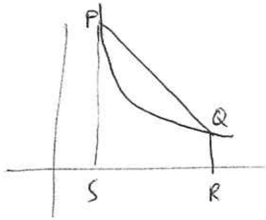
$= c^2q^2 + \left[\frac{cp - cp - cq}{pq}\right]^2$

$= c^2q^2 + \frac{c^2}{p^2}$

$= (d_{AP})^2$

$\therefore d_{AP} = d_{BQ}$

(iii) Area = trapezium PQRS $- \int_{cp}^{cq} \frac{c^2}{x}\,dx$



$= \frac{1}{2}c(q-p)\left[c\left(\frac{1}{p} + \frac{1}{q}\right)\right] - c^2\int_{cp}^{cq} \frac{1}{x}\,dx$

$= \frac{1}{2}c^2(q-p)\frac{(q+p)}{pq} - c^2\left[\ln x\right]_{cp}^{cq}$

$= \frac{1}{2}c^2\frac{(q^2-p^2)}{pq} - c^2(\ln cq - \ln cp)$

$= \underline{c^2(q^2-p^2)} + c^2\ln\left(\frac{p}{q}\right)$

$\ln cq - \ln cp$

$= \ln\frac{cq}{cp}$

$= -\ln\frac{p}{q}$

4(b) (i)

$$\begin{array}{r} x^2 + 3x - 9 \\ x^2+2 \overline{\smash{)}\, x^4 + 3x^3 - 7x^2 + 11x - 1} \\ -(x^4 \quad + 2x^2) \\ \hline 3x^3 - 9x^2 \\ -(3x^3 \quad + 6x) \\ \hline -9x^2 + 5x \\ -(-9x^2 \quad - 18) \\ \hline 5x + 17 \end{array}$$

$\therefore P(x) = (x^2+2)[x^2+3x-9] + 5x + 17$

(ii) x^2+2 divides into the first 3 terms as above, but the algorithym gets modified with the $2x$:

$$\begin{array}{r} 2x \\ -(\quad +6x) \\ \hline -9x^2 - 4x \\ -(-9x^2 \quad - 18) \\ \hline -4x + 18 \end{array} \leftarrow \text{Remainder.}$$

$\therefore ax + b \equiv 4x - 18$ to ensure the remainder cancels.

alternatively: use result

$x^4 + 3x^3 - 7x^2 + 11x - 1 = (x^2+2)(x^2+3x-9) + 5x + 17$

$x^4 + 3x^3 - 7x^2 + 11x - 1 - 5x - 18 = (x^2+2)(x^2+3x-9)$

$x^4 + 3x^3 - 7x^2 + 6x - 18 = (x^2+2)(x^2+3x-9)$

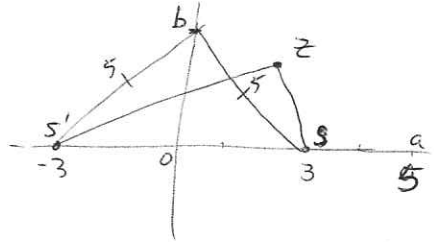
$b = -18$

$2x + ax = 6x$

$a = 4$

4(c)

(i) $|z-3| + |z+3| = 10$



$ZS + ZS' = 2a$ by definition.

$= 10$

$\therefore a = 5$

$b^2 = 25 - 9$ (from ΔOSb)

$= 16$

$\therefore b = 4$

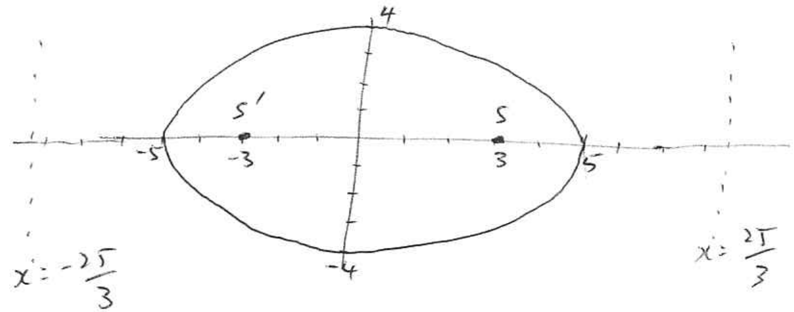
Hence $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

(ii) Directrices $x = \pm \dfrac{a}{e}$ $\left(e = \dfrac{3}{5}\right)$

$\therefore x = \pm 5 \times \dfrac{5}{3}$

$= \pm \dfrac{25}{3}$

foci $(\pm 3, 0)$



Q5

(a) $u_1 = 1 \quad u_2 = 5 \quad u_n = 5u_{n-1} - 6u_{n-2}$

Prove $u_n = 3^n - 2^n$

For $n=3$: $u_3 = 5u_2 - 6u_1$

$= 5(5) - 6(1)$

$= 19$

1. Show true for $n=3$

$u_3 = 3^3 - 2^3$

$= 27 - 8 = 19$

$\therefore$ true for $n=3$

2. Assume true for $n=k$

i.e given $u_k = 5u_{k-1} - 6u_{k-2}$

then $u_k = 3^k - 2^k$

3. Prove true for $n=k+1$

i.e given $u_{k+1} = 5u_k - 6u_{k-1}$

show that $u_{k+1} = 3^{k+1} - 2^{k+1}$

$LHS = 5u_k - 6u_{k-1}$

$= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$

$= 5(3^k - 2^k) - [2 \cdot 3 \cdot 3^{k-1}] + [3 \cdot 2 \cdot 2^{k-1}]$

$= 5 \cdot 3^k - 5 \cdot 2^k - 2 \cdot 3^k + 3 \cdot 2^k$

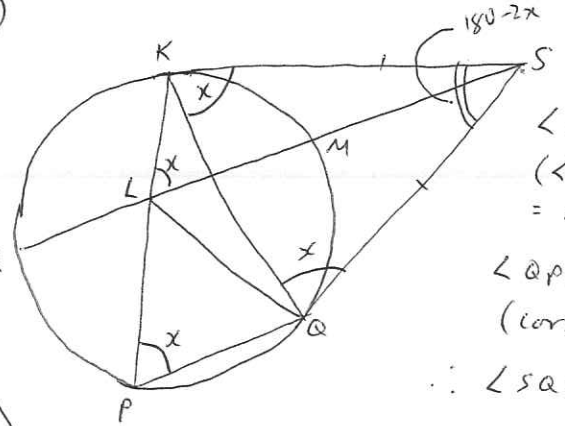
$= 3 \cdot 3^k - 2 \cdot 2^k$

$= 3^{k+1} - 2^{k+1}$

$= RHS$

Q 5 (b)

(i)



$\angle SQK = \angle QPK = x$

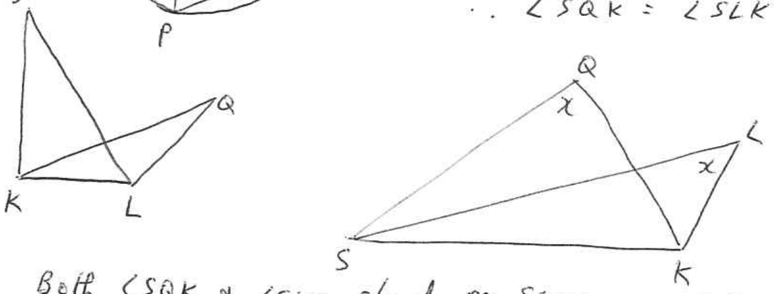
($\angle$ between tangent & chord = $\angle$ in alt segment).

$\angle QPK = \angle SLK$

(corresponding $\angle$'s in // lines =)

$\therefore \angle SQK = \angle SLK$

(ii)



Both $\angle SQK$ & $\angle SLK$ stand on same arc KS of circle LKSQ

(iii) ΔKSQ is isosceles with $KS = QS$ $\therefore \angle SKQ = \angle SQK = x$

in ΔKSQ, $\angle KSQ = 180 - 2x$ ($\angle$ sum of Δ)

$\therefore$ in cyclic quad ($\angle KSQ$) $\angle KLQ = 2x$ (suppl with $\angle KSQ$).

now $\angle KLQ$ is ext $\angle$ of ΔPLQ.

$\therefore \angle PQL = x$

$\therefore \Delta PQL$ is isosceles

$\therefore LQ = LP$ (sides opp = $\angle$'s).

5(c)

$$\sum_{r=0}^{n} (-1)^n \frac{{}^nC_r}{x+n} = \frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

if $x=1$:

$$\sum_{r=0}^{n} (-1)^n \frac{{}^nC_r}{1+n} = \frac{n!}{1(2)(3)\cdots(n+1)}$$

$$\text{RHS} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\text{LHS} = \frac{(-1)^0\,{}^nC_0}{1} + \frac{(-1)^1\,{}^nC_1}{2} + \frac{(-1)^2\,{}^nC_2}{3} + \dots + \frac{(-1)^n\,{}^nC_n}{1+n}$$

$$= 1 + \frac{-{}^nC_1}{2} + \frac{{}^nC_2}{3} + \dots + \frac{(-1)^n}{1+n}$$

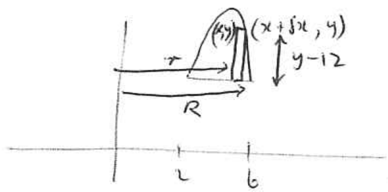
i.e. $1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \dots \frac{(-1)^n}{1+n} = \frac{1}{n+1}$

(d) $12 = 8x - x^2$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

POI: $(2,12)$ & $(6,12)$



X-sect Area of shell

$= \pi(R^2 - r^2)$

$= \pi((x+\delta x)^2 - x^2)$

$= \pi(x^2 + 2x\delta x + \delta x^2 - x^2)$

$= \pi\, 2x\delta x$ (ignoring δx^2)

$= 2\pi x \delta x$

Vol of typical shell

$\delta V = 2\pi x \delta x (y-12)$

$\therefore$ total volume

$$= 2\pi \int_2^6 x(8x - x^2 - 12)\,dx$$

$$= 2\pi \int_2^6 8x^2 - x^3 - 12x \,dx$$

$$= 2\pi \left[\frac{8x^3}{3} - \frac{x^4}{4} - 6x^2\right]_2^6$$

$$= 2\pi \left[\left(\frac{8\times 6^3}{3} - \frac{6^4}{4} - 6\times 36\right) - \left(\frac{8\times 2^3}{3} - \frac{2^4}{4} - 6\times 4\right)\right]$$

Q6

a(i) $\dfrac{A(\pi - 2x) + Bx}{x(\pi - 2x)} = \dfrac{1}{x(\pi - 2x)}$

$\frac{1}{\pi}(\pi - 2x) + Bx = 1$

$1 - \frac{2x}{\pi} + Bx = 1$

$Bx = \frac{2x}{\pi}$

$\therefore B = \frac{2}{\pi}$

(ii) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{\pi}\frac{1}{x} + \frac{1}{\pi}\left(\frac{2}{\pi - 2x}\right)$

$= \frac{1}{\pi}\ln x - \frac{1}{\pi}\ln(\pi - 2x)$

$= \frac{1}{\pi}\left[\ln x - \ln(\pi - 2x)\right]$

$= \frac{1}{\pi}\left[\ln \frac{x}{\pi - 2x}\right]_{\pi/6}^{\pi/3}$

$= \frac{1}{\pi}\left[\ln \frac{\pi/3}{\pi/3} - \ln \frac{\pi/6}{4\pi/6}\right]$

$= \frac{1}{\pi}\left[0 - \ln \frac{1}{4}\right]$

$= \frac{1}{\pi}\left[-\ln 2^{-2}\right]$

$= \frac{1}{\pi}\left[2\ln 2\right]$

$= \frac{2}{\pi}\ln 2$

(iii) $\int_a^b f(x)\,dx$ Let $u = a+b-x \Rightarrow du = -dx$

$x = a+b-u$ $\therefore dx = -du$

when $x = a$, $u = b$

$x = b$, $u = a$

$$\therefore \int_a^b f(x)\,dx = \int_b^a f(a+b-u)\,-du$$

$$= \int_a^b f(a+b-u)\,du$$

$$= \int_a^b f(a+b-x)\,dx$$

iv) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi-2x)} = \int \frac{\cos^2\left[\frac{\pi}{6}+\frac{\pi}{3}-x\right]\,dx}{\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\left[\pi-2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]}$$

$$= \int \frac{\cos^2\left(\frac{\pi}{2}-x\right)\,dx}{\left(\frac{\pi}{2}-x\right)\left[\pi-2\left(\frac{\pi}{2}-x\right)\right]}$$

$$= \int \frac{\sin^2 x}{\left(\frac{\pi}{2}-x\right)(2x)}\,dx$$

$$= \int \frac{1-\cos^2 x}{x(\pi-2x)}$$

So, $$\int \frac{\cos^2 x\,dx}{x(\pi-2x)} = \int \frac{1}{x(\pi-2x)} - \frac{\cos^2 x}{x(\pi-2x)}\,dx$$

$$\therefore 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi-2x)} = \frac{2}{\pi}\ln 2$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi-2x} = \frac{1}{\pi}\ln 2$$

(15)

6 (b)(i) $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ & $y^2 = \frac{a^2}{4}\left[e^{2x/a} + 2 + e^{-2x/a}\right]$ (16)

$$\frac{dy}{dx} = \frac{a}{2}\left[\frac{1}{a}e^{x/a} - \frac{1}{a}e^{-x/a}\right]$$

$$= \frac{1}{2}\left[e^{x/a} - e^{-x/a}\right]$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left[e^{2x/a} - 2 + e^{-2x/a}\right]$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2x/a}}{4} - \frac{1}{2} + \frac{e^{-2x/a}}{4}$$

$$= \frac{1}{4}\left[e^{2x/a} + 2 + e^{-2x/a}\right]$$

$$= \frac{y^2}{a^2}$$

(ii) $$S = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x \sqrt{\frac{y^2}{a^2}}\,dx$$

$$= \int_0^x \frac{y}{a}\,dx$$

$$= \frac{1}{2}\int_0^x e^{x/a} + e^{-x/a}\,dx$$

$$= \frac{1}{2}\left[a e^{x/a} - a e^{-x/a}\right]_0^x$$

$$= \frac{a}{2}\left[e^{x/a} - e^{-x/a}\right] - \frac{a}{2}\left[a - a\right]$$

$$\therefore S = \frac{a}{2}\left[e^{x/a} - e^{-x/a}\right]$$

and $$y^2 - a^2 = \frac{a^2}{4}\left[e^{x/a} + e^{-x/a}\right]^2 - a^2$$

$$= \frac{a^2}{4}\left[e^{2x/a} + 2 + e^{-2x/a} - 4\right]$$

$$= \frac{a^2}{4}\left[e^{2x/a} - 2 + e^{-2x/a}\right]$$

$$= \frac{a^2}{4}\left(e^{x/a} - e^{-x/a}\right)^2$$

$$= S^2$$

$$\therefore S = \sqrt{y^2 - a^2}$$

Q7

a(i) $f(x) = \sqrt{2-\sqrt{x}}$ $\quad x \geq 0$ for $\sqrt{x}$

$x \leq 4$ for $\sqrt{2-\sqrt{4}}$

(ii) $f(x) = (2 - x^{\frac{1}{2}})^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}x^{-\frac{1}{2}}\right)\left(2-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}$

$= \dfrac{-1}{4\sqrt{x}\,\sqrt{2-\sqrt{x}}} < 0$ for $0 \leq x \leq 4$

$\therefore$ a decreasing function.

$x = 0 \quad f(x) = \sqrt{2}$

$x = 4 \quad f(x) = 0$

Range $0 \leq y \leq \sqrt{2}$

(iii) $u = 2 - \sqrt{x}$

$du = -\frac{1}{2}x^{-\frac{1}{2}}\,dx$

$= \dfrac{-1}{2\sqrt{x}}\,dx$

$\therefore dx = -2\sqrt{x}\,du$

$= -2(2-u)\,du$

when $x = 4$, $u = 0$

$x = 0$, $u = 2$

$\therefore \int_0^4 \sqrt{2-\sqrt{x}}\,dx$

$= -2\int_2^0 \sqrt{u}\,(2-u)\,du$

$= 2\int_0^2 2u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= 2\left[\dfrac{4u^{\frac{3}{2}}}{3} - \dfrac{2u^{\frac{5}{2}}}{5}\right]_0^2$

$= 2\left[\left(\frac{4}{3}2^{\frac{3}{2}} - \frac{2}{5}2^{\frac{5}{2}}\right) - (0-0)\right]$

$= 2\left(\frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2}\right) = \dfrac{32\sqrt{2}}{15}$

7(b) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$

$= \int_0^{\frac{\pi}{4}} \tan^2 x \tan^{n-2} x\,dx$

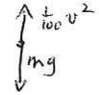
$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\tan^{n-2} x\,dx$

$= \int \underbrace{\sec^2 x}_{f'(x)}\,\underbrace{\tan^{n-2} x}_{f(x)}\,dx - \int \tan^{n-2} x\,dx$

$= \left[\dfrac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$

$I_n = \left(\dfrac{1}{n-1} - 0\right) - I_{n-2}$

$\therefore I_n + I_{n-2} = \dfrac{1}{n-1}$

(c)(i) 

$F = m\ddot{x} = mg - \frac{1}{100}v^2 \qquad m = 1$

$\therefore \ddot{x} = g - \frac{1}{100}v^2$

(ii) V_T occurs when $\ddot{x} = 0$

i.e. $0 = g - \dfrac{V_T^2}{100}$

$V_T^2 = 100g$

$V_T = 10\sqrt{g}$

(iii) $\ddot{x} = g - \dfrac{v^2}{100}$ $\quad$ we want $v = f(x)$

$v\dfrac{dv}{dx} = g - \dfrac{v^2}{100}$

$= \dfrac{100g - v^2}{100}$

$\therefore \dfrac{dv}{dx} = \dfrac{100g - v^2}{100v} \Rightarrow \dfrac{dx}{dv} = \dfrac{100v}{100g - v^2}$

$x = \int_0^v \dfrac{100v}{100g - v^2}\,dv$

$x = -50\int_0^v \dfrac{-2v}{100g - v^2}\,dv$

$= \left[-50\ln\left|100g - v^2\right|\right]_0^v$

$\therefore x = -50\ln(100g - v^2) + 50\ln(100g)$ $\quad$ now write $v = f(x)$

$= 50\ln\dfrac{100g}{100g - v^2}$

$\dfrac{x}{50} = \ln\dfrac{100g}{100g - v^2}$

$e^{\frac{x}{50}} = \dfrac{100g}{100g - v^2}$

$\dfrac{100g - v^2}{100g} = e^{-\frac{x}{50}}$

$v^2 = 100g - 100g\,e^{-\frac{x}{50}}$

$= 100g\left(1 - e^{-\frac{x}{50}}\right)$

$= V_T^2\left(1 - e^{-\frac{x}{50}}\right)$

8(a)(i) $x^2 + 2xy + y^5 = 4$

$2x + 2y + \dfrac{dy}{dx}2x + 5y^4\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}\left[2x + 5y^4\right] = -2x - 2y$

$\therefore \dfrac{dy}{dx} = \dfrac{-2x - 2y}{2x + 5y^4}$

$\therefore$ @ $P(X, Y)$, $\dfrac{dy}{dx} = \dfrac{-2X - 2Y}{2X + 5Y^4}$

(8) (b) (i) $\tan 4\theta = 1 \qquad 0 \le \theta \le \pi$

$0 \le 4\theta \le 4\pi$

$4\theta = \frac{\pi}{4}, \ \pi + \frac{\pi}{4}, \ 2\pi + \frac{\pi}{4}, \ 3\pi + \frac{5\pi}{4}$

$4\theta = \frac{\pi}{4}, \ \frac{5\pi}{4}, \ \frac{9\pi}{4}, \ \frac{13\pi}{4}$

$\theta = \frac{\pi}{16}, \ \frac{5\pi}{16}, \ \frac{9\pi}{16}, \ \frac{13\pi}{16}$

(ii) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

(iii) Let $t = \tan\theta$ $\quad \tan 4\theta = \dfrac{2\tan 2\theta}{1-\tan^2 2\theta}$

$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2} \times \frac{(1-t^2)^2}{(1-t^2)^2}$$

$$= \frac{4t(1-t^2)}{(1-t^2)^2-4t^2} = \frac{4t-4t^3}{1-2t^2+t^4-4t^2}$$

$\therefore \tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$

(iv) Let $\tan 4\theta = 1$ and $\tan\theta = x$

$1 = \dfrac{4x-4x^3}{1-6x^2+x^4}$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

solns are $x = \tan\theta$ where θ is soln to $\tan 4\theta = 1$

$\therefore$ solns are $\tan\frac{\pi}{16}, \ \tan\frac{5\pi}{16}, \ \tan\frac{9\pi}{16}, \ \tan\frac{13\pi}{16}$.

(v) Product of roots $= +\frac{e}{a}$ in $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$= 1$

(vi) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^2\left(x^2 + 4x - 6 - \frac{4}{x} + \frac{1}{x^2}\right) = 0$

$x^2 \ne 0 \quad \left(\left(x^2 + \frac{1}{x^2}\right) + 4\left(x - \frac{1}{x}\right) - 6\right) = 0$

$\left(\left(x - \frac{1}{x}\right)^2 + 2 + 4\left(x - \frac{1}{x}\right) - 6\right) = 0$

$\left(\left(x - \frac{1}{x}\right)^2 + 4\left(x - \frac{1}{x}\right) - 4\right) = 0$

Let $u = x + \frac{1}{x}$ $\quad u^2 + 4u - 4 = 0$

$u = \dfrac{-4 \pm \sqrt{16 + 4 \times 4}}{2} = \dfrac{-4 \pm 4\sqrt{2}}{2}$

$= -2 \pm 2\sqrt{2}$

$x - \frac{1}{x} = -2 \pm 2\sqrt{2}$

smallest value is $\tan\frac{\pi}{16} - \dfrac{1}{\tan\frac{\pi}{16}} = -2 - 2\sqrt{2}$

$\therefore \tan\frac{\pi}{16} - \cot\frac{\pi}{16} = -2 - 2\sqrt{2}$

(pg)

(20)

(ii) horiz @ P suggests $-2X - 2y = 0$

$X = -y$

i.e $X^2 + 2X(-X) + (-X)^5 = 4$

$X^2 - 2X^2 - X^5 - 4 = 0$

i.e $X^5 + X^2 + 4 = 0$

(iii) Let $y = X^5 + X^2 + 4$

$\frac{dy}{dx} = 5X^4 + 2X$

Let $\frac{dy}{dx} = 0 \quad 2X + 5X^4 = 0$

$X(2 + 5X^3) = 0$

$X = 0$ or $X^3 = \frac{-2}{5}$

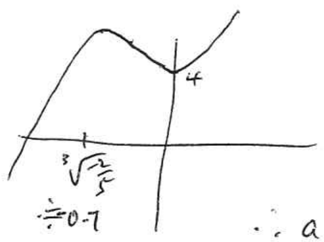
$X = \sqrt[3]{\frac{-2}{5}}$

$\frac{d^2y}{dx^2} = 20X^3 + 2$

@ $X = 0$ $\frac{d^2y}{dx^2} = 2 > 0 \therefore$ U min.

at $x = \sqrt[3]{\frac{-2}{5}}$ $\quad \frac{d^2y}{dx^2} = 20\left(\sqrt[3]{\frac{-2}{5}}\right)^3 + 2 < 0$ ∩ max

$\therefore$ min at $(0, 4)$ & max at $x = \sqrt[3]{\frac{-2}{5}}$



when $x = -1$, $y = -1 + 1 + 4 > 0$

$x = -2$, $y = -32 + 4 + 4 < 0$

$\therefore$ a unique real solution occurs between -1 & -2.