2011 EXTENSION 1 TRIAL SOLUTIONS

QUESTION 1

(a) $A(-3,6)$ $B(7,-4)$ $P(1,2)$

$k:1$

$$1 = \frac{-3\times 1 + 7\times k}{k+1}$$

$$k+1 = -3+7k$$

$$6k = 4$$

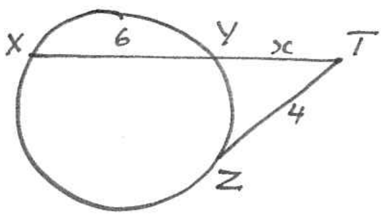
$$k = 2/3$$

(b) $\frac{d}{dx}(\overset{u}{x}\,\overset{v}{\tan^{-1}x})$

$$= x \times \frac{1}{1+x^2} + 1\times \tan^{-1}x$$

$$= \frac{x}{1+x^2} + \tan^{-1}x$$

(c)



$$x(x+6) = 4^2$$

$$x^2+6x-16=0$$

$$(x+8)(x-2)=0$$

$$x=-8 \text{ or } 2 \qquad x>0$$

$$\therefore\ YT = 2\text{ cm}$$

(d) $\int_1^{\sqrt2}\frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt2}$

$$= \sin^{-1}\frac{\sqrt2}{2} - \sin^{-1}\frac12$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$= \frac{\pi}{12}$$

(e) $\int_{-2}^{2} x\sqrt{x+2}\,dx$ let $u = x+2$, $du = dx$; $x=2, u=4$; $x=-2, u=0$

$$= \int_0^4 (u-2)\times\sqrt{u}\,du$$

$$= \int_0^4 \left(u^{3/2} - 2u^{1/2}\right)du$$

$$= \left[\frac25 u^{5/2} - 2\times\frac23 u^{3/2}\right]_0^4$$

$$= \frac25\times 4^{5/2} - \frac43\times 4^{3/2} - (0-0)$$

$$= 2/5\times 32 - 4/3\times 8$$

$$= 2\,2/15$$

Q2 (a) (i) $LHS = \frac{1+\cos 2A}{\sin 2A}$

$$= \frac{1+\cos^2A - \sin^2A}{2\sin A\cos A}$$

$$= \frac{\cos^2A + \cos^2A}{2\sin A\cos A}$$

$$= \frac{\cancel{2}\cos^{\cancel{2}}A}{\cancel{2}\sin A\cancel{\cos A}}$$

$$= \cot A$$

$$= RHS$$

(ii) $\cot \pi/12 = \frac{1+\cos\pi/6}{\sin\pi/6}$

$$= \frac{1+\frac{\sqrt3}{2}}{\frac12}\times\frac22$$

$$= \frac{2+\sqrt3}{1}$$

$$= 2+\sqrt3$$

(b) $Vol = \int_0^{\pi}\pi\times\cos^2\frac{x}{2}\,dx$

$$\cos 2A = 2\cos^2A - 1$$

$$\therefore\ \cos^2A = \tfrac12(1+\cos 2A)$$

$$\cos^2\frac{x}{2} = \tfrac12(1+\cos x)$$

$$\therefore\ Vol = \frac{\pi}{2}\int_0^{\pi}(1+\cos x)\,dx$$

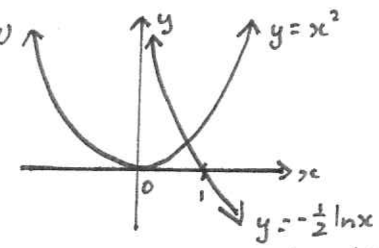
$$= \frac{\pi}{2}\left[x+\sin x\right]_0^{\pi}$$

$$= \frac{\pi}{2}[\pi+\sin\pi] - \frac{\pi}{2}[0+\sin 0]$$

$$= \frac{\pi}{2}(\pi+0) - \frac{\pi}{2}(0+0)$$

$$= \frac{\pi^2}{2}\text{ square units}$$

(c) (i)



(ii) curves meet at $x^2 = -\frac12\ln x$

$\therefore$ let $f(x) = x^2 + \frac12\ln x$

$$f'(x) = 2x + \frac{1}{2x}$$

$f(0.5) = \left(\frac12\right)^2 + \frac12\ln(0.5) \doteqdot -0.09657\ldots$ $\quad f'(0.5) = 2\times\frac12 + \frac{1}{2\times\frac12} = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{-0.09657\ldots}{2}$$

$$= 0.54828\ldots$$

$$\doteqdot 0.55$$

Q3 (a) $P(x) = x^3 - 3x^2 - 4x + 12$

(i) $P(-2) = (-2)^3 - 3(-2)^2 - 4(-2) + 12$

$= -8 - 12 + 8 + 12$

$= 0$

$\therefore x+2$ is a factor of $P(x)$

(ii) $x^3 - 3x^2 - 4x + 12 = (x+2)(x^2 + Mx + 6)$

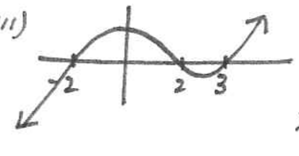
coefficient of x^2 : $-3 = 2 + M$

$\therefore M = -5$

$\therefore P(x) = (x+2)(x^2 - 5x + 6)$

$= (x+2)(x-3)(x-2)$

(iii)



$P(x) \le 0$

$x \le -2$ or $2 \le x \le 3$

(b) $(2+x)(1-x)^5$

coeff. of x^3 comes from

$2 \times {}^5C_3\,(-x)^3 + x \times {}^5C_2\,(-x)^2$

coefficient $= 2 \times {}^5C_2 \times (-1)^3 + {}^5C_2 \times 1$

$= {}^5C_2(-2+1)$

$= \frac{5 \times 4}{2 \times 1} \times -1$

$= -10$

(c) (i) $2x(1-x)$ has max. at

$x = \frac{0+1}{2}$

$= \frac{1}{2}$

max value $= 2 \times \frac{1}{2}(1 - \frac{1}{2})$

$= \frac{1}{2}$

(ii) $\sin^{-1}(0 \times 1) = 0$

$\sin^{-1}(1 \times 0) = 0$

$\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

$\therefore$ Range is $0 \le y \le \frac{\pi}{6}$

Q4

(a) $x^3 - 6x^2 + 10x - 4 = 0$

let roots be $\alpha - d, \alpha, \alpha + d$

sum of roots : $3\alpha = 6$

$\therefore \alpha = 2$

product of roots : $\alpha(\alpha - d)(\alpha + d) = 4$

$\alpha(\alpha^2 - d^2) = 4$

$2(4 - d^2) = 4$

$4 - d^2 = 2$

$d^2 = 2$

$d = \pm\sqrt{2}$

$\therefore x = 2 - \sqrt{2},\ 2,\ 2 + \sqrt{2}$

(b) $v = a + be^{-kx}$

(i) $x = 0,\ v = 2a$

$2a = a + be^0$

$b = a$

$\therefore v = a + ae^{-kx}$

(ii) $v = a(1 + e^{-kx})$

since $e^{-kx} > 0$ for all x,

$1 + e^{-kx}$ is always positive

$\therefore$ if $a > 0$, $v > 0$ always

OR, if $a < 0$, $v < 0$ always

$\therefore$ The particle never changes direction.

Q4 continued

(c)(i) Show $4(1^3+2^3+3^3+\dots+n^3)=n^2(n+1)^2$

if $n=1$, $LHS = 4\times 1^3$

$= 4$

$RHS = 1^2(1+1)^2$

$= 1\times 2^2$

$= 4$

$= LHS$

$\therefore$ Result is true for $n=1$

Assume the result is true for $n=k$ (a +ve integer)

ie assume $4(1^3+2^3+3^3+\dots+k^3)=k^2(k+1)^2$ ①

Prove result is true for $n=k+1$

ie Prove $4(1^3+2^3+3^3+\dots+k^3+(k+1)^3)=(k+1)^2(k+2)^2$

$LHS = 4(1^3+2^3+\dots+k^3)+4(k+1)^3$

$= k^2(k+1)^2+4(k+1)^3$ using ①

$= (k+1)^2[k^2+4(k+1)]$

$= (k+1)^2[k^2+4k+4]$

$= (k+1)^2(k+2)^2$

$= RHS$

$\therefore$ If result is true for $n=k$ then it will also be true for $n=k+1$.

$\therefore$ By mathematical induction, result is true for all integers $n \geq 1$

(ii) $\lim_{n\to\infty} \dfrac{(1^3+2^3+3^3+\dots+n^3)}{n^4}$

$= \lim_{n\to\infty} \dfrac{\frac{\cancel{n^2}(n+1)^2}{4}}{\cancel{n^4}\,n^2}$

$= \lim_{n\to\infty} \dfrac{n^2+2n+1}{4n^2}$

$= \lim_{n\to\infty} \dfrac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$

$= \dfrac{1+0+0}{4}$

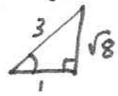
$= \dfrac{1}{4}$

5 (a) $\tan\{\cos^{-1}(-\tfrac{1}{3})\}$

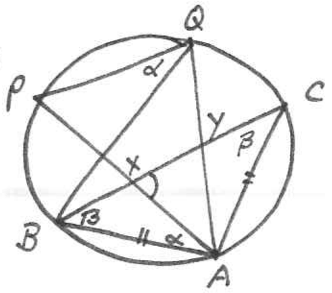
$= \tan\{\pi-\cos^{-1}(\tfrac{1}{3})\}$

$= -\tan\{\cos^{-1}(\tfrac{1}{3})\}$ (2nd quadrant)

$= -\sqrt{8}$



(b)



(i) $\angle AXC = \alpha+\beta$ (exterior $\angle$ of ΔABX)

(ii) $\angle PQB = \angle PAB = \alpha$ ($\angle^s$ in same segment)

(iii) $\angle BCA = \angle ABC = \beta$ (base $\angle^s$ of isosceles ΔABC)

$\therefore \angle AQB = \angle BCA = \beta$ ($\angle^s$ in same segment)

(iv) $\angle PQB + \angle AQB = \alpha+\beta$

$= \angle AXC$

$\therefore$ PQYX is a cyclic quad. since its exterior $\angle$ is equal to its opposite interior $\angle$)

(c) $\dfrac{d^2x}{dt^2} = 10x-2x^3$ $\quad v=0, x=1$

(i) $a = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

$\therefore \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 10x-2x^3$

$\tfrac{1}{2}v^2 = 5x^2-\dfrac{x^4}{2}+C$

$v=0, x=1 \quad \therefore 0 = 5-\tfrac{1}{2}+C$

$C = -4\tfrac{1}{2}$

$\therefore \tfrac{1}{2}v^2 = 5x^2-\dfrac{x^4}{2}-4\tfrac{1}{2}$

$v^2 = 10x^2-x^4-9$

(ii) particle at rest when $v=0$

$\therefore 10x^2-x^4-9=0$

$x^4-10x^2+9=0$

$(x^2-9)(x^2-1)=0$

$x^2 = 9$ or 1

$\therefore x = \pm 3$ or ± 1 These are the positions where the particle is at rest.

Q6 a) (i) If line is a tangent, there is only 1 point of intersection.

$$x = K(y-1) + \frac{2}{K} \quad ①$$

$$x^2 = 8(y-1) \Rightarrow y-1 = \frac{x^2}{8} \text{ into } ①$$

$$x = K \times \frac{x^2}{8} + \frac{2}{K}$$

$$8x = Kx^2 + \frac{16}{K}$$

$$K^2x^2 - 8Kx + 16 = 0$$

$$(Kx-4)^2 = 0$$

$$Kx - 4 = 0$$

$$x = \frac{4}{K}$$

$\therefore$ only 1 pt of intersection

$\therefore$ line is a tangent

(ii) lines from (5, 4) are:

$$5 = m(4-1) + \frac{2}{m}$$

$$5m = 3m^2 + 2$$

$$3m^2 - 5m + 2 = 0$$

$$(3m-2)(m-1) = 0$$

$$\therefore m = \frac{2}{3} \text{ or } 1$$

$\therefore$ gradients of tangents are $\frac{1}{m} = \frac{3}{2}$ and 1

let $\theta = \angle$ between 2 lines

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{1 - 3/2}{1 + 1 \times 3/2}\right|$$

$$= \frac{1/3}{5/3}$$

$$= 1/5$$

$$\therefore \theta \doteqdot 11^\circ$$

b) $P(2ap, ap^2)$ $Q(2aq, aq^2)$ $x^2 = 4ay$

(i) grad. of PQ: $\dfrac{aq^2 - ap^2}{2aq - 2ap} = \dfrac{\cancel{a}(\cancel{q-p})(q+p)}{2\cancel{a}(\cancel{q-p})}$

$$= \frac{p+q}{2}$$

eq'n of PQ: $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$$y - ap^2 = \frac{p+q}{2}x - \cancel{2}ap\left(\frac{p+q}{\cancel{2}}\right)$$

$$y - ap^2 = \frac{p+q}{2}x - ap^2 - apq$$

$$\therefore y = \frac{p+q}{2}x - apq$$

(ii) focal chord passes through (0, a)

$$\therefore a = \frac{p+q}{2} \times 0 - apq$$

$$a = -apq$$

$$\therefore pq = -1$$

(iii) $M = \left(\dfrac{2ap + 2aq}{2}, \dfrac{a(p^2+q^2)}{2}\right)$

$$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$$

$$N = (a(p+q), -a)$$

$$\therefore T = \left(a(p+q), \frac{\frac{a(p^2+q^2)}{2} - a}{2}\right)$$

$$= \left(a(p+q), \frac{a(p^2+q^2) - 2a}{4}\right)$$

$$= \left(a(p+q), \frac{a(p^2+q^2-2)}{4}\right)$$

(iv) From (iii) $p + q = \frac{x}{a}$ ①

and $p^2 + q^2 = \frac{4y}{a} + 2$ ②

also, from (ii) $pq = -1$ ③

subst. ①, ②, ③ into $(p+q)^2 = p^2 + q^2 + 2pq$

$$\left(\frac{x}{a}\right)^2 = \frac{4y}{a} + 2 + 2 \times -1$$

$$\frac{x^2}{a^2} = \frac{4y}{a}$$

$$\therefore x^2 = 4ay$$

Q7 (a) (i) $x = 20t \Rightarrow t = \frac{x}{20}$ ①

$y = 20\sqrt{3}\,t - 5t^2$ ②

Subst. ① into ②

$y = 20\sqrt{3}\left(\frac{x}{20}\right) - 5\left(\frac{x}{20}\right)^2$

$y = \sqrt{3}x - \frac{5x^2}{400}$

$\therefore y = \sqrt{3}x - \frac{x^2}{80}$ ③

(ii) $y = \frac{1}{4}x$ is equation of the hill ④

Subst ④ into ③

$\frac{1}{4}x = \sqrt{3}x - \frac{x^2}{80}$

$\frac{x^2}{80} - \sqrt{3}x + \frac{1}{4}x = 0 \qquad \times 80$

$x^2 + 20x - 80\sqrt{3}x = 0$

$\therefore x(x + 20 - 80\sqrt{3}) = 0$

$x = 0$ or $80\sqrt{3} - 20 \qquad x > 0$

$\therefore x = (80\sqrt{3} - 20)$

(iii) Subst. $x = 80\sqrt{3} - 20$ into $y = \frac{1}{4}x$

$y = 20\sqrt{3} - 5$

$OA^2 = x^2 + y^2$

$= (80\sqrt{3} - 20)^2 + (20\sqrt{3} - 5)^2$

$OA = 122.213\ldots$

$\doteqdot 122$ metres

(iv) at A, $x = 80\sqrt{3} - 20 = 20t$

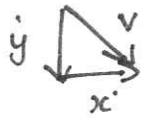
$\therefore t = 4\sqrt{3} - 1$

$\dot{x} = 20$

$\dot{y} = 20\sqrt{3} - 10t$

at A, $\dot{y} = 20\sqrt{3} - 10(4\sqrt{3} - 1)$

$= -20\sqrt{3} + 10$



$v^2 = (\dot{x})^2 + (\dot{y})^2$

$= (20)^2 + (-20\sqrt{3} + 10)^2$

$v = 31.736\ldots$

$\therefore$ speed $\doteqdot 32$ m/s at A

(b) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$ ①

let $x = 1 \therefore 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ ②

let $x = -1 \therefore 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ ③

② − ③ $\quad 2^n = 2\times\binom{n}{1} + 2\times\binom{n}{3} + \cdots + 2\times\binom{n}{n-1}$

$\div 2 \quad \frac{2^n}{2} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1}$

$\therefore \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1} = 2^{n-1}$