Student Number:____________________

Student Name:____________________

Teacher's Name:____________________

ABBOTSLEIGH

AUGUST 2011
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided with this paper.
- All necessary working should be shown in every question.

Total marks – 84
- Attempt Questions 1-7.
- All questions are of equal value.
- Answer each question in a separate writing booklet.

Outcomes assessed

Preliminary course

- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives which require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving binomials and projectiles
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Harder applications of the Mathematics course are included in this course. Thus the Outcomes from the Mathematics course are included.

Outcomes from the Mathematics course

Preliminary course

- **P2** provides reasoning to support conclusions that are appropriate to the context
- **P3** performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities
- **P4** chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques
- **P5** understands the concept of a function and the relationship between a function and its graph
- **P6** relates the derivative of a function to the slope of its graph
- **P7** determines the derivative of a function through routine application of the rules of differentiation
- **P8** understands and uses the language and notation of calculus

HSC course

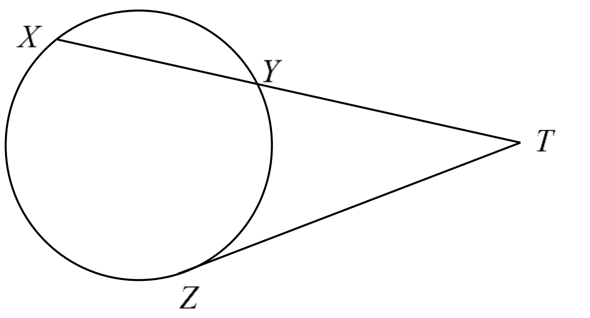
- **H2** constructs arguments to prove and justify results
- **H3** manipulates algebraic expressions involving logarithmic and exponential functions
- **H4** expresses practical problems in mathematical terms based on simple given models
- **H5** applies appropriate techniques from the study of calculus, geometry, trigonometry and series to solve problems
- **H6** uses the derivative to determine the features of the graph of a function
- **H7** uses the features of a graph to deduce information about the derivative
- **H8** uses techniques of integration to calculate areas and volumes
- **H9** communicates using mathematical language, notation, diagrams and graphs

Question 1 (12 marks) Start a New Booklet. **Marks**

(a) The point $P(1,2)$ divides the interval AB in the ratio $k:1$. If A is the point $(-3,6)$ and B is the point $(7,-4)$, find the value of k. 2

(b) Find $\frac{d}{dx}(x\tan^{-1}x)$ 2

(c)



Not to scale

ZT is a tangent to a circle. XYT is a secant intersecting the circle at X and Y.
Given that $ZT = 4$ cm and $XY = 6$ cm, find the length of YT. 2

(d) Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}}$ 3

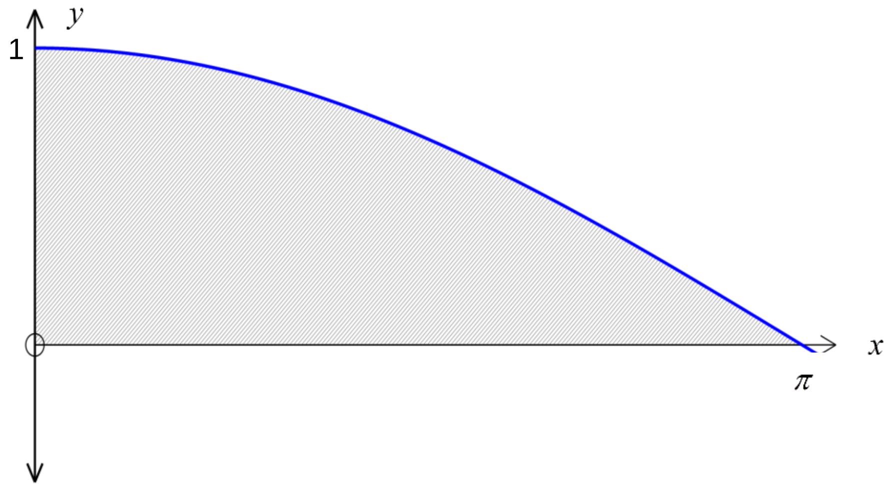
(e) Use the substitution $u = x+2$ to evaluate $\displaystyle\int_{-2}^{2} x\sqrt{x+2}\,dx$ 3

Question 2 (12 marks) Start a new booklet. **Marks**

(a) (i) Show that $\dfrac{1+\cos 2A}{\sin 2A} = \cot A$ 2

(ii) Hence find the exact value of $\cot\dfrac{\pi}{12}$. 2

(b) The diagram below shows the graph of $y = \cos\frac{1}{2}x$ for $0 \le x \le \pi$.



The shaded area is rotated about the $x-$axis. Find the volume of the solid formed. 3

(c) (i) Show by means of a sketch, that the curves $y = x^2$ and $y = -\frac{1}{2}\ln x$ meet at a single point. 2

(ii) By taking 0.5 as a first approximation to the root of $x^2 + \frac{1}{2}\ln x = 0$, use Newton's method once to find a better approximation of where the two curves meet. Give your answer correct to 2 decimal places. 3

Question 3 (12 marks) Start a new booklet. **Marks**

(a) Consider the polynomial $P(x) = x^3 - 3x^2 - 4x + 12$.

(i) Show that $(x+2)$ is a factor of $P(x)$. 1

(ii) Hence, or otherwise, express $P(x)$ as a product of three linear factors. 2

(iii) Solve $P(x) \le 0$. 2

(b) Find the coefficient of x^3 in the expansion of $(2+x)(1-x)^5$ 3

(c) (i) Find the maximum value of $2x(1-x)$. 2

(ii) Hence, or otherwise, find the range of the function given by
$f(x) = \sin^{-1}\{2x(1-x)\}$ in the interval $0 \le x \le 1$. 2

Question 4 (12 marks) Start a new booklet. **Marks**

(a) Solve the cubic equation $x^3 - 6x^2 + 10x - 4 = 0$ given that the roots form an arithmetic series. 3

(b) The velocity v at a position x of a particle moving in a straight line, is given $v = a + be^{-kx}$ where a, b and k are constants.

(i) Find b if $v = 2a$ at the origin. 1

(ii) Show that the particle never changes its direction of motion. 2

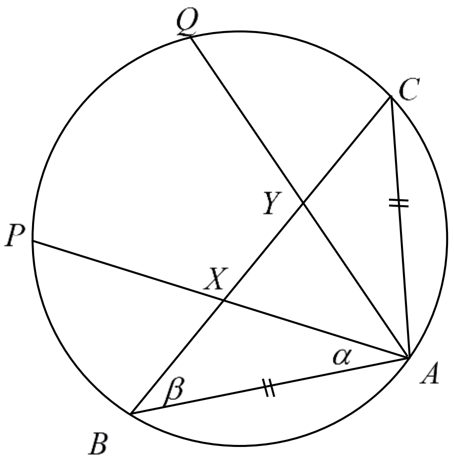
(c) (i) Show that $4\left(1^3 + 2^3 + 3^3 + ... + n^3\right) = n^2\left(n+1\right)^2$ for all integers $n \geq 1$, using the principle of mathematical induction. 3

(ii) Hence evaluate $\lim_{n \to \infty} \left\{ \frac{1^3 + 2^3 + 3^3 + ... + n^3}{n^4} \right\}$ 3

Question 5 (12 marks) Start a new booklet. **Marks**

(a) Evaluate $\tan\left\{\cos^{-1}\left(\frac{-1}{3}\right)\right\}$. 2

(b) In the circle below, $AB = AC$. Let $\angle PAB = \alpha$ and $\angle ABC = \beta$.



Not to Scale

(i) Copy the diagram onto your answer booklet and state why $\angle AXC = \alpha + \beta$ 1

(ii) Give a reason why $\angle PQB = \alpha$. 1

(iii) Prove $\angle AQB = \beta$. 1

(iv) Prove $XYQP$ is a cyclic quadrilateral. 2

(c) A particle P moves in a straight line so that its distance from O after t secs is x metres. The acceleration of P from O is given by the equation $\frac{d^2x}{dt^2} = 10x - 2x^3$. The particle is at rest 1 metre to the right of O.

(i) Find v^2 in terms of x, where v is the velocity of the particle. 3

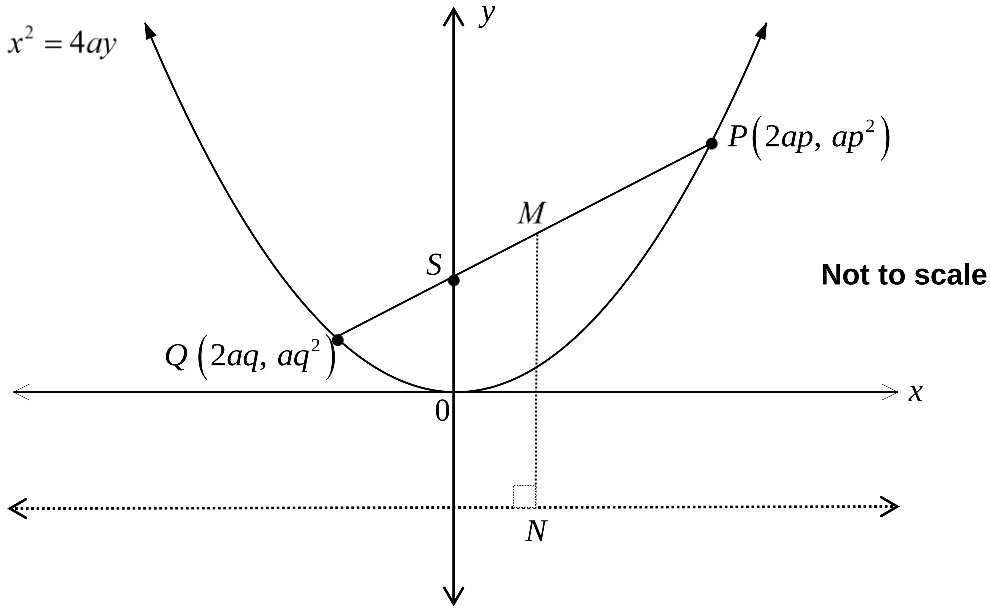
(ii) Hence find all positions where the particle is at rest. 2

Question 6 (12 marks) Start a new booklet. **Marks**

(a) (i) Show that the line $x = k(y-1) + \dfrac{2}{k}$ is a tangent to the parabola $(y-1) = \dfrac{x^2}{8}$ for all values of k, $(k \neq 0)$. 3

(ii) Hence, or otherwise, find the acute angle between the tangents drawn to this parabola from the point $(5,4)$. 2

(b)



The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the equation of the chord PQ is $y = \dfrac{p+q}{2}x - apq$. 2

(ii) Show that if the chord PQ passes through the focus $S(0, a)$ then $pq = -1$. 1

(iii) M is the midpoint of the focal chord PQ. N lies on the directrix such that MN is perpendicular to the directrix. T is the midpoint of MN.

Show that the coordinates of T are $\left(a(p+q), \dfrac{a(p^2+q^2-2)}{4} \right)$. 2

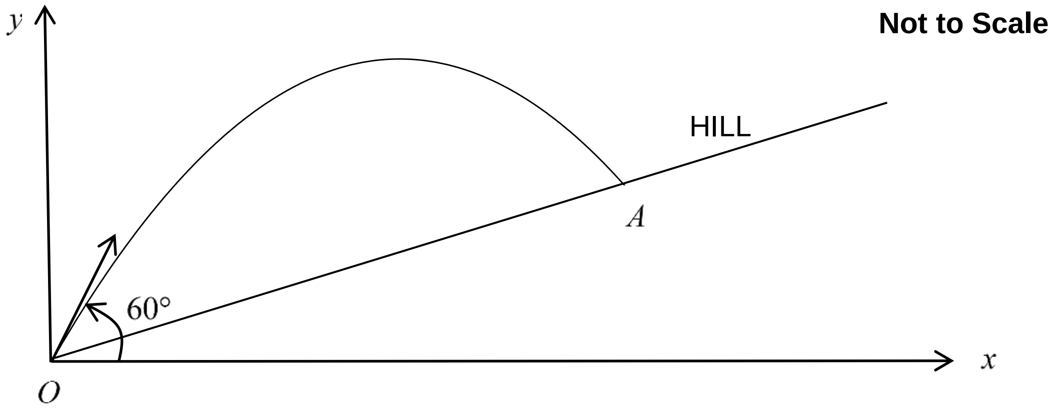
(iv) Find the equation of the locus of T. 2

Question 7 (12 marks) Start a new booklet. **Marks**

(a) An arrow is thrown from the ground with initial velocity $40\,m\,s^{-1}$ at an angle of $60°$ to the horizontal. The position of the arrow at time t seconds is given by the parametric equations:

$$x = 20t$$
$$y = 20\sqrt{3}t - 5t^2$$

where $10\,ms^{-2}$ is the acceleration due to gravity. (You are NOT required to derive these)



(i) Show that the Cartesian equation of the path of the arrow is given by $y = \sqrt{3}x - \dfrac{x^2}{80}$. 2

(ii) The arrow is thrown above a hill with a gradient of $\frac{1}{4}$. Show that the horizontal distance travelled by the projectile when it lands on the hill (the x-coordinate of A) is $\left(80\sqrt{3} - 20\right)$ metres. 2

(iii) Hence find the distance OA up the hill the from the point of projection of the arrow to the point of landing. Give your answer to the nearest metre. 2

(iv) Find the speed of the arrow when it lands on the hill at the point A. Give your answer to the nearest $m\,s^{-1}$. 3

(b) Consider the binomial expansion $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$ where n is an even number.

Prove that $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ... + \binom{n}{n-1} = 2^{n-1}$ 3

END OF PAPER