BOARD OF SENIOR SCHOOL STUDIES

MATHEMATICS SYLLABUS

AND

NOTES TO THE SYLLABUS

YEAR 11 AND YEAR 12

4 UNIT COURSE

(Unit value of 4 Unit Course in Year 11 is 3 Units)

(Approved by the Board on 6th February, 1980)

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BOARD OF SENIOR SCHOOL STUDIES

MATHEMATICS SYLLABUS

4 UNIT COURSE

The Board recognises that the aims and objectives of the syllabus may be achieved in a variety of ways and by the application of many different techniques. Success in the achievement of these aims and objectives is the concern of the Board which does not, however, either stipulate or evaluate specific teaching methods.

INTRODUCTION

The Mathematics 4 Unit Course is defined in the same terms as the 3 Unit Course in other subjects. Thus it offers a suitable preparation for study of the subject at tertiary level, as well as a deeper and more extensive treatment of certain topics than is offered in other Mathematics Courses.

This syllabus is designed for students with a special interest in mathematics who have shown that they possess special aptitude for the subject. It represents a distinctly high level in school mathematics involving the development of considerable manipulative skill and a high degree of understanding of the fundamental ideas of algebra and calculus. These topics are treated in some depth. Thus the course provides a sufficient basis for a wide range of useful applications of mathematics as well as an adequate foundation for the further study of the subject.

AIMS AND OBJECTIVES

The general aim is to present mathematics as a living art which is intellectually exciting, aesthetically satisfying, and relevant to a great variety of practical situations.

Specific aims of the course are:

(a) To offer a programme which will be of interest and value to pupils with the highest levels of mathematical ability at the stage of the Higher School Certificate and which will present some challenge to such pupils.

(b) To study useful and important mathematical ideas and techniques appropriate to these levels of ability.

(c) To develop both an understanding of these ideas and techniques and an ability to apply them to the study and solution of a wide variety of problems.

(d) To provide the mathematical background necessary for further studies in mathematics, and useful for concurrent study of subjects such as science, economics and industrial arts.

SYLLABUS

PART I

IA. THE MATHEMATICS 3 UNIT COURSE

The whole of the syllabus for the 3 Unit course is included in the 4 Unit course.

IB. ADDITIONAL EXAMPLES IN GEOMETRIC APPLICATIONS OF DIFFERENTIATION

Extension of the ideas and techniques of the 3 Unit topic on Geometrical Applications of Differentiation to rational functions, logarithmic and exponential functions, the circular functions, and simple combinations of such functions.

PART II

1. COMPLEX NUMBERS

(a) Origin of the idea of a complex number in connection with the existence of solutions of quadratic equations. Exploration of the consequences of introducing a symbol , satisfying $i^2 = -1$. Argand diagram, modulus, argument, conjugate. Geometric representation of addition and multiplication of complex numbers. The relations

$$|z_1 + z_2| \le |z_1| + |z_2|, \; |z_1 z_2| = \quad .$$

$$\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}, \; \bar{z}_1 \cdot \bar{z}_2 = \overline{z_1 z_2},$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}.$$

(b) The identity $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$, where n is an integer. The roots of ± 1.

(c) Square roots of complex numbers.

(d) Curves and regions in the complex plane determined by simple arithmetic operations on complex numbers.

2. ALGEBRA

(a) *Polynomials as functions*

Polynomials with real coefficients, polynomial equations. Division of one polynomial by another. Factors and roots, the remainder and factor theorems. Relations between the degree and the number of roots.

Statement (without proof) of the fundamental theorem of algebra. Use to give the standard factorisations of a real polynomial over the complex field and the real field.

General relations between roots and coefficients.

(b) *Partial fractions*

Practical methods for finding partial fractions for rational functions whose denominators have *simple* linear or quadratic factors. (This is to be applied to examples in integration.)

3. FURTHER CALCULUS

(a) (i) Calculation of the derivative for simple implicit functions, and its use in sketching curves.

(ii) The conic sections:- defining equations, curve sketching, and identification of foci and directrix. Parametrizations of the circle, the ellipse, the parabola and the hyperbola.

(b) Integration:

(i) Use of change of variable.

(ii) Integration by parts.

(iii) Simple rational functions.

(iv) Applications of the above techniques.

(c) Computing of areas and volumes - some examples by the "slice" technique.

4. ELEMENTARY PARTICLE DYNAMICS

(a) Kinematics and dynamics of a particle in one and two dimensions.

(b) Motion in a circle.

(c) Free motion in a resistive medium under gravity.

NOTES TO THE SYLLABUS

PART I

A. The Notes on the 3 Unit Course apply unless they restrict the range of a topic treated elsewhere in this Syllabus. It is emphasised that the 3 Unit Course is an integral part of the 4 Unit Course. The 4 Unit Course is to give students experience with harder problems based on the 3 Unit Course, and it is anticipated that such problems would be included in any 4 Unit Course examination.

B. Examples of additional types of functions to be covered are $\frac{x}{1+x}$, $\frac{x}{1+x^2}$, xe^{-x}, e^{-x^2}, $x \log x$, $\frac{\sin x}{x}$, $x \sec x$, $x + \sin x$.

Here, as in the treatment of this topic in the 3 Unit Course, emphasis is to be placed on the relation between the geometrical representation of a function and properties of the function. For example, from the formula

$$f(x) = \frac{x}{1+x^2} \quad (x \in \mathbb{R})$$

the facts that $f(x)$ is odd and has the same sign as x, behaves like $g(x) = x$ for x near 0 and like $h(x) = \frac{1}{x}$ for $|x|$ large, has (at least) a maximum in $x > 0$ and hence (at least) an inflexion in $x > 0$, should be observed before f' or f'' are calculated. The geometrical significance of each of these observations should be understood.

Similarly, simple properties of a derivative should be observed and used to deduce facts about its (possibly unknown) primitive function. For example, if

$$f'(x) = e^x(x^2 - 2x - 3) \; ,$$

and $f(x)$ tends to 0 as x tends to $-\infty$, then f is increasing (and hence positive) until it obtains a maximum at $x = -1$, it decreases to a minimum at $x = 3$, then increases and tends to $+\infty$ as x tends to $+\infty$.

These ideas require understanding and use of the sign of a function and its derivatives, use of all the rules for differentiation, and an understanding of asymptotes.

PART II

1. COMPLEX NUMBERS

(a) The intention is to familiarise students with the basic notations for, arithmetic operations on, and geometric representation of complex numbers, which are regarded as an extension of the real numbers by adjoining a symbol i, satisfying $i^2 = -1$. The arithmetic of numbers $a + ib$ is to obey the usual rules. Real and imaginary parts, modulus, argument and conjugate are defined and the appropriate notations introduced. Use Re(z) and Im(z) for real and imaginary parts of z. Argand diagram representation clarifies the meaning of the arithmetic operations and the new terms.

(b) The identity for $\cos n\theta + i \sin n\theta$ is proved by induction on n for n a positive integer, and then extended to negative integers. The complex nth roots of ± 1 are obtained in modulus-argument form, and plotted on the Argand diagram.

(c) Square roots of complex numbers are found by solving the relevant real equations. ($z^2 = w$ gives $x^2 - y^2 + 2ixy = u + iv$, which gives two equations for x, y in terms of u, v. In modulus-argument form the relevant equations are $|z|^2 = |w|$, $2 \arg z = \arg w$/)

(d) Typical curves and regions are those defined by simple equations or inequalities, such as $\mathrm{Re}(z) = \frac{1}{2}$, $|z| = 3$, $|z - i| \leq 5$, $\mathrm{Im}(z) > 2$, $0 < \arg z < \pi/2$, and extending to examples such as the following (Q5 (ii) of the 1977 H.S.C. 4 Unit Second Paper):

Describe, in geometric terms, the locus (in the Argand plane) represented by

$$2|z| = z + \bar{z} + 4.$$

2. ALGEBRA

It is expected that pupils will be familiar with the commonly used algebraic properties of integers, rational numbers and real numbers. A distinction should be drawn between the integers (in which divisibility properties are important and lead to the ideas of prime, factor and division with remainder) and the rational or real numbers (where divisibility by any non-zero number is possible). Decimal representations of real (and rational numbers) should be discussed briefly. The words *integral domain* and *field* might be introduced and related to the various number systems studied. Standard notations for the common number systems are:

Z for the (ring of) integers 0, ±1, ±2....,
Q for the (field of) rational numbers,
R for the (field of) real numbers,
C for the (field of) complex numbers,

and it is suggested that these symbols be introduced and used.

(a) *Polynomials as functions*

A polynomial function of a real variable, say

$$P(x) = a_0 + a_1x + \ldots + a_nx^n \quad (a_0, \ldots, a_n \text{ real numbers})$$

assigns to each real number b in its domain the number

$$P(b) = a_0 + a_1b + \ldots + a_nb^n.$$

If the coefficients of P are integers, then P(b) is an integer for each integer value of b. P may also be considered then as a function from the set of integers into the same set. Similarly, if the coefficients of P are rational numbers, then P defines a function from the field of rationals into the same field. Since the real field may be considered as a subset of the complex field, a real polynomial (i.e., one with real coefficients) also defines a function from the complex field into itself. The domain of a polynomial function P is important, for it may determine the number of roots P has. For example, if

$$P(x) = x^2 - 2,$$

then P(x) = 0 has no solutions in the rational field, but two solutions in the real field. If

$$P(x) = x^2 + 2,$$

then P(x) = 0 has no solutions in the real field, but two solutions in the complex field.

Addition and multiplication of real polynomial functions are defined as for all functions of a real variable. The properties of real numbers, together with the index laws, imply that the sum or product of two polynomial functions is again a polynomial function. Because of this, we may apply to polynomial functions the ideas of divisibility and factorisation.

The division transformation is established, yielding the polynomial identity

$$P(x) \equiv A(x)Q(x) + R(x),$$

where the quotient Q(x) and the remainder R(x) are polynomials, and deg R < deg A (possibly R is the zero polynomial in which all coefficients are zero). The use of the 'identically equals' sign ≡ means here that the coefficients of the corresponding powers of x on each side are equal.

When A(x) is of the first degree, say A(x) = x - a, then

$$P(x) \equiv (x-a)Q(x) + r,$$

where either r is the zero polynomial or deg r = 0, so that in either case r is a constant independent of x. Evaluating this expression for the value x = a,

$$P(a) = r,$$

a result called the remainder theorem. The case r = 0 gives the factor theorem: x-a is a factor of P(x) if and only if P(a) = 0.

Repeated application of the factor theorem to P(x) yields a linear factor (x-a) for each root a of P(x) in the domain of P. In particular, if P(x) has degree n and if P has n distinct roots $a_1, \ldots, a_n$, then

$$P(x) \equiv c(x-a_1)\ldots(x-a_n)$$

for some constant c. This means that if $b \neq a_i$ for each i, then $P(b) \neq 0$, and consequently a real polynomial function of degree n > 0 cannot have more than n distinct real roots. The argument used no properties of the real numbers other than that they form a field, and the result applies to polynomials over any field. Earlier remarks on the significance of the domain for the number of roots apply now to the number of linear factors.

It may happen that $P(x)=(x-a)Q(x)$ and that $(x-a)$ is a factor of $Q(x)$ (and hence that $x-a$ is a root of $Q(x)$). The number a is then called a repeated or multiple root of $P(x)$. If

$$P(x)=(x-a)^r S(x),$$

where r is a positive integer and $S(a)\neq 0$, then a is called a root of $P(x)$ of order or multiplicity r, and $(x-a)^r$ is called a factor of $P(x)$ of order r. A simple root or factor is one of order 1. By convention, a root of order r is counted as r (equal) roots (each root is counted "according to its multiplicity"). With this convention, the argument above relating roots and factors extends to cases of multiple roots, and the result is that a real polynomial of degree $n\geqslant 0$ cannot have more than n roots, where each distinct root is counted according to its multiplicity. In the case where $P(x)$ is monic (i.e., the coefficient of the highest power of x is 1) of degree $n>1$ and has n linear factors $x-a_1,\ldots,x-a_n$, we obtain

$$P(x) \equiv (x-a_1)\ldots(x-a_n)$$
$$\equiv (x^n - (\Sigma a_i)x^{n-1} + (\Sigma a_i a_j)x^{n-2} - \ldots + (-1)^n a_1\ldots a_n.$$

Comparing coefficients on each side yields the relations between the roots of P and its coefficients.

From the meaning of $\equiv$, it is clear that if P and Q are identical polynomials, then $P(x)$ and $Q(x)$ are the same function: $P(x) = Q(x)$ for every x. The result above, that a real polynomial function of degree n cannot have more than n roots, implies that for the real or complex fields, if $P(x)$ and $Q(x)$ define the same function, then $P(x) \equiv Q(x)$, i.e., P and Q are identical polynomials. For if not, then $P-Q$ is not the zero polynomial, so $\deg(P-Q) = n \geq 0$. Hence $P(x) - Q(x) = 0$ for at most n values of x. Choose a real number b distinct from these values of x. Then $P(b) - Q(b) \neq 0$, or $P(b) \neq Q(b)$, showing that P and Q do not define the same function.

It should now be proved that if two real polynomials of degrees at most n take the same values at $n + 1$ distinct points, then they are identical.

(It might be pointed out that some of the above results are false for polynomials over other domains. For example, the numbers 0, 1, 2, 3, 4 form a field F under the operations of modulo 5 arithmetic. The polynomials x and x^5, considered as functions on F into F, are the *same* function, but $x^5 \not\equiv x$. The numbers 0, 1, 2, 3, 4, 5 form a ring R under the operations of modulo 6 arithmetic. The polynomial $x^2 + x$, of degree 2, has the 4 roots 0, 2, 3, 5 in R.)

The "fundamental theorem of algebra" asserts that every polynomial $P(x)$ of degree $n > 0$ over the complex field has at least one root. Using this result, the factor theorem should now be used to prove (by induction on the degree) that a polynomial of degree $n > 0$ with real (or complex) coefficients has exactly n complex roots (each counted according to its multiplicity) and is expressible as a product of exactly n complex linear factors.

The relations between roots and coefficients derived above are therefore valid for every such polynomial, and hence for polynomials with real coefficients (which nevertheless may have no real roots). The fact that complex roots of real polynomials occur in conjugate pairs leads directly to the factorisation of real polynomials over the real field as a product of real linear and real quadratic factors.

The formation of a polynomial whose roots are a given multiple of the roots of a given polynomial, or whose roots differ from the roots of a given polynomial by a given constant, should be known.

(b) *Partial fractions*

The theory in 2(a) above has led to the result that any real polynomial is expressible as a product of real linear and real quadratic factors, the latter factors having no real roots. The same polynomial, considered over the complex field, is expressible as a product of (real or complex) linear factors.

These factorisations may be used to simplify the problem of integration of rational functions, by means of the so-called partial fraction decomposition of rational functions. (A formal development of the theory, based on the Euclidean algorithm for polynomials, is not required. The intention is for pupils to become familiar with practical procedures for treating the simplest cases. The general theory, which proves the existence and nature of the partial fraction decomposition of any rational function, or techniques for treating repeated linear or quadratic factors could both be developed with interested students (especially if appropriate examples are met in exercises on integration), but are both excluded from the syllabus.)

By definition, a rational function $f(x)$ is the ratio of two polynomials:

$$f(x) = A(x)/B(x),$$

defined for all values of x except those for which $B(x) = 0$. If $\deg A \geq \deg B$, we divide B into A,

$$A(x) = B(x)Q(x) + R(x), \quad \deg R < \deg B,$$

obtaining

$$f(x) = Q(x) + R(x)/B(x).$$

The problem of partial fraction decomposition arises when $B(x)$ is a product of polynomials of lower degree, say $B(x) = B_1(x)B_2(x)$ with $\deg B_1 > 0$, $\deg B_2 > 0$. Can we find polynomials $m_1(x)$, $m_2(x)$ such that

$$\frac{R(x)}{B(x)} = \frac{m_1(x)}{B_1(x)} + \frac{m_2(x)}{B_2(x)} \ ?$$

This will be so if

$$R(x) = m_1(x)B_2(x) + m_2(x)B_1(x) \ ,$$

and comparison of degrees shows that we may suppose $\deg m_1 < \deg B_1$, $\deg m_2 < \deg B_2$. Rather than discuss the general theory, we confine attention to the following useful cases.

(i) B(x) is a product of distinct linear factors:

$$B(x) = c(x-a_1)\dots(x-a_n).$$

In this case, we wish to discover if constants $c_1,\dots,c_n$ exist so that

$$\frac{R(x)}{B(x)} = \frac{c_1}{x-a_1} + \dots + \frac{c_n}{x-a_n}.$$

Consider the case when $n = 2$. Then $R(x) = cx + d$, and on multiplying by $B(x)$, we seek c_1, c_2 so that

$$cx + d = c_1(x-a_2) + c_2(x-a_1).$$

Comparing coefficients,

$$c = c_1 + c_2$$

$$d = -a_2c_1 - a_1c_2.$$

Since $a_1 \neq a_2$, these equations may be solved for c_1 and c_2. In the general case, the fact that $a_1,\dots,a_n$ are distinct enables the coefficient equations to be solved for $c_1,\dots,c_n$.

Other methods may be used to find $c_1,\dots,c_n$, which are often quicker to use. For example, if we multiply the expression for $R(x)/B(x)$ by $x-a_1$, and note that $B(a_1) = 0$, the result may be written

$$R(x) \cdot \frac{x-a_1}{B(x)-B(a_1)} = c_1 + (x-a_1)\frac{c_2}{x-a_2} + \dots + \frac{c_n}{x-a_n}.$$

Let x tend to a_1. The right-hand side tends to c_1, the left-hand side tends to $R(a_1)/B'(a_1)$. Thus

$$c_1 = R(a_1)/B'(a_1)$$

and the method enables all the c_i to be found.

(ii) B(x) is a product of distinct linear factors and a simple quadratic factor. Here the decomposition is of the form

$$\frac{R(x)}{B(x)} = \frac{c_1}{x-a_1} + \dots + \frac{c_n}{x-a_n} + \frac{dx+e}{x^2+bx+c}, \quad \deg R < n + 2,$$

and again the numbers $c_1,\dots,c_n$, d, e may be found by comparing coefficients, or by combining that method with others. For example, $c_1,\dots,c_n$ may be found by the method described immediately above. If none of $a_1,\dots,a_n$ is 0, putting $x = 0$ gives e. Multiplying by x and letting x tend to infinity gives d. If say $a_1 = 0$, first find d, and then e may be found by selecting a small integer value for x, distinct from $a_1,\dots,a_n$, to give a simple equation for e.

3. FURTHER CALCULUS

(a) (i) The discussion is to be confined to examples such as simple polynomials in x and y where y is given implicitly as a function of x, or vice-versa.

(ii) Cartesian equations in standard form ($\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$) are taken as the defining equations for ellipse and hyperbola, whose curves should be sketched, and their variations examined as the ratio $\frac{a}{b}$ varies. The parametric representations

$$x = a\cos\theta\ ,\ y = b\sin\theta \qquad \text{(ellipse)}$$

$$x = a\sec\theta\ ,\ y = b\tan\theta \qquad \text{(hyperbola)}$$

$$x = at\ ,\ y = a/t \qquad \text{(rectangular hyperbola)}$$

are useful in graphing these curves in order to obtain their shapes.

The intention is to acquaint candidates with the basic shapes of these curves, their common equations, the focus-directrix properties, and the equations of tangents and normals in cartesian and parametric forms. The focus-directrix properties relate them to the parabola. The fact that these curves are sections of a cone might be discussed, but there is no need to attempt to derive their equations from this description.

Locus problems involving tangents, normals or chords are restricted to the circle, parabola and rectangular hyperbola.

(b) The intention here is to provide further practice in the evaluation of simple integrals, and to introduce the useful techniques of substitution and integration by parts. (The relationships of these to the corresponding differentiation rules should be pointed out, but no formal justification is required. The basic rules should be stated carefully.)

(i) The substitutions to be treated are simple, e.g. $x = \sin\theta$, $v = x^2$, $t = \tan\frac{\theta}{2}$, and applied to simple integrands (e.g. $\int\sqrt{(1-x^2)}\,dx$, $\int x(1+x^2)^n dx$, $\int \frac{\sin\theta}{2+\sin\theta}\,d\theta$). The effect on limits of integration is required, and definite integrals are to be treated.

(ii) The work on integration by parts should include the intergrands $\sin^{-1}x$, $e^{ax}\cos bx$, $\log x$, $x^n \log x$ (n an integer) and should be extended to particular examples of recurrence relations (e.g. $\int x^n e^x dx$, $\int_0^{\pi/2} \cos^n x\,dx$). (Recurrence relations such as $\int_0^1 x^m(1-x)^n dx$, which involve more than one integer parameter, are excluded.)

(iii) The rational functions treated should not require partial fraction decompositions more complicated than those discussed in 2(b) above.

(iv) The above techniques may be required in the solution of problems on other topics in the syllabus

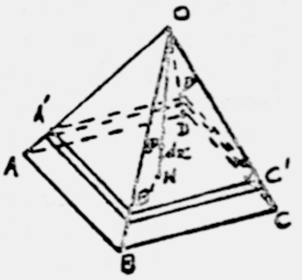
(c) This is intended to provide further practice in the finding of definite integrals where the process of "limiting sum" has an intuitive geometrical representation, which should be presented in each example.

The evaluation of infinite series by this technique is excluded, as is the evaluation of integrals by summation of series. Typical problems to be considered are:

(a) volumes of cones on bases whose areas are known or can be calculated (so that areas of "slices" can be found by similarity).

Example: Find the volume of a pyramid of height h on a square base of side length a.

Let O be the vertex of the pyramid, and let N be the point where the perpendicular, from O to the base ABCD, meets the base, so ON = h. Let P on ON be distant x from O.



The plane through P, parallel to the base, meets OA, OB, OC, OD respectively at A', B', C', D'. By similar triangles,

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{OD'}{OD} = \frac{x}{h},$$

hence

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{x}{h}.$$

Thus the area of the square A'B'C'D' is

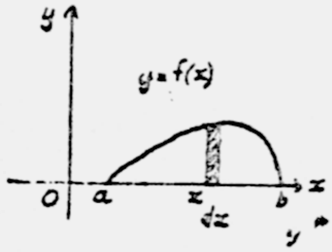
$A'B'.B'C' = \frac{x^2}{h^2} AB.BC = \frac{x^2}{h^2} a^2$. Hence a thin slice of thickness dx on the base A'B'C'D' has volume $\frac{x^2 a^2}{h^2} dx$.

The volume of the pyramid is therefore

$$\int_0^h \frac{x^2 a^2}{h^2} dx = \frac{a^2}{h^2} \int_0^h x^2 dx = \frac{1}{3} a^2 h.$$

(b) The volume of a solid of revolution by using cylindrical "slices".

Example: The continuous function f(x) satisfies f(a) = f(b) = 0, where 0 < a < b, and is positive for a < x < b. The area between y = f(x) and the x-axis, for a ≤ x ≤ b, is rotated about the y axis. Prove that the volume of the solid thus obtained is



$$2\pi \int_a^b xy \; dx \; .$$

Consider a thin cylindrical slice whose base is the annulus of radii x and x + dx, of height y = f(x).

Volume of slice = area of base x height
= $2\pi x dx$ x y.

Hence Volume of solid = $\int_a^b 2\pi xydx$.

4. ELEMENTARY PARTICLE DYNAMICS

Students should be able to represent mathematically motions described in physical terms, and should be able to explain in physical terms features given by mathematical descriptions of motions in one or two dimensions. Emphasis should be placed upon making physical interpretations of mathematical concepts such as range, domain, and stationary or extreme values of functions, rather than upon intricate formal manipulations.

(a) The classical statement of Newton's First and Second laws of motion should be given as an illustration of the application of mathematics to the physical world. Resolution of forces, accelerations and velocities in horizontal and vertical directions is to be used to obtain the appropriate equations of motion in two dimensions.

(b) The notions of *angular velocity* and *centripetal force* should be understood.

(c) Discussion of resisted motion should be restricted to simple examples that can be solved explicitly, including the case of a particle moving vertically under gravity and subject to a resistance proportional to a power of the speed.

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