HSC Mathematics 2008 Solutions

Question 1 :

a) $2\cos\left(\frac{\pi}{5}\right) = \underline{1.62}$ (3 sig figs)

b) $3x^2 + x - 2 = \underline{(3x-2)(x+1)}$

c) $\frac{2}{n} - \frac{1}{n+1} = \frac{2(n+1)-n}{n(n+1)} = \frac{n+2}{n(n+1)}$

d) $|4x-3| = 7$

$\therefore\ 4x - 3 = 7$ or $4x - 3 = -7$

$4x = 10$ $\quad$ $4x = -4$

$\underline{x = 2\tfrac{1}{2}}$ $\quad$ $\underline{x = -1}$

e) $(\sqrt{3}-1)(2\sqrt{3}+5) = 2\times 3 + 5\sqrt{3} - 2\sqrt{3} - 5$

$= \underline{1 + 3\sqrt{3}}$

f) $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ with $a = 3,\quad d = 4,\quad n = 21$

$S_{21} = \frac{21}{2}[6 + 20\times 4] = \underline{903}$

Question 2 :

a) (i) $\frac{d}{dx}(x^2+3)^9 = 9(x^2+3)^8 \times 2x = \underline{18x(x^2+3)^8}$

(ii) $\frac{d}{dx}(x^2 \ln x) = 2x.\ln x + x^2.\left(\frac{1}{x}\right) = \underline{2x\ln x + x}$

(iii) $\frac{d}{dx}\left(\frac{\sin x}{x+4}\right) = \frac{(x+4)\cos x - \sin x}{(x+4)^2}$

b) midpoint is given by $\left(\frac{-1+5}{2}, \frac{4+8}{2}\right)$

$\therefore\ M \equiv \underline{(2\,,\,6)}$

Equation of line with gradient $-\frac{1}{2}$ is :

$y - 6 = -\frac{1}{2}(x-2)$

$2y - 12 = -x + 2$

$\underline{x + 2y - 14 = 0}$

c) (i) $\int \frac{1}{x+5}\,dx = \underline{\ln(x+5) + c}$

(ii) $\int_0^{\frac{\pi}{12}} \sec^2 3x\,dx = \left[\frac{1}{3}\tan 3x\right]_0^{\frac{\pi}{12}} = \frac{1}{3}\times 1 - 0$

$= \frac{1}{3}$

Question 3 :

a) (i) gradient of BC , $m_1 = \dfrac{5-3}{1-0} = 2$

gradient of AD : $2x-y-1=0 \Rightarrow y=2x-1$

$\therefore \; m_2 = 2$

since $m_1 = m_2$ then $BC \parallel AD$

$\therefore$ $ABCD$ is a trapezium with a pair of opposite sides parallel.

(ii) For D, substitute 5 for y in $2x-y-1=0$.

<u>ie</u> $2x-5-1=0 \Rightarrow x=3$

$\therefore$ <u>D is (3, 5)</u>

(iii) BC = $\sqrt{(1-0)^2+(5-3)^2} = \sqrt{5}$

(iv) $d = \left|\dfrac{2\times0-1\times3-1}{\sqrt{2^2+1^2}}\right| = \left|\dfrac{-4}{\sqrt{5}}\right| = \dfrac{4}{\sqrt{5}}$

(v) AD = $\sqrt{(3-0)^2+(5+1)^2} = \sqrt{45} = 3\sqrt{5}$

Area of trapezium = $\frac{1}{2}\left(\sqrt{5}+3\sqrt{5}\right)\times\dfrac{4}{\sqrt{5}} = \dfrac{4\cancel{\sqrt{5}}\times4}{2\times\cancel{\sqrt{5}}}$ = <u>8 units^2</u>

b) (i) $\dfrac{d}{dx}\left(\ln(\cos x)\right) = \dfrac{-\sin x}{\cos x} = -\tan x$

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \tan x \, dx = \left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}} = -\ln\left(\frac{1}{\sqrt{2}}\right)+\ln 1$

$= \underline{\frac{1}{2}\ln 2}$

Question 4 :

a) $\angle YRX = \angle XRQ$ (given)

$\angle YXR = \angle XRQ$ (alternate angles, $XY \parallel QR$)

$\therefore \; \angle YRX = \angle YXR$

$\therefore$ the base angles of ΔXYR are equal and ΔXYR is isosceles.

b) (i) height = $50\times(1\cdot2)^8$ after 8 zoom applications.

= <u>215 mm</u> (nearest mm)

(ii) need $50\times(1\cdot2)^n > 400 \Rightarrow (1\cdot2)^n > 8$

$\therefore \; n.\ln(1\cdot2) > \ln 8$

$n > 11\cdot405$

The least number of zoom applications needed is <u>12 times</u>.

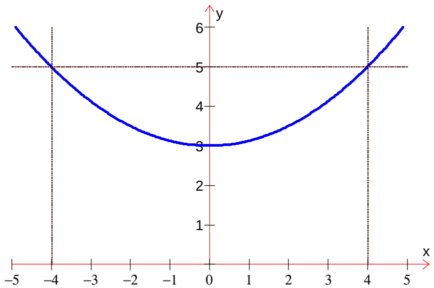
c) (i) vertex is (0, 3)

(ii) focal length is 2 units $\Rightarrow$ focus is (0,5)

(iii) $x^2 = 8(y-3)$

becomes

$y = \dfrac{x^2}{8}+3$



(iv) Area = area of rectangle – area under parabola (-4<x<4)

$= 5\times8 - 2\times\displaystyle\int_0^4\left(\frac{x^2}{8}+3\right)dx$

$= 40 - 2\times\left[\dfrac{x^3}{24}+3x\right]_0^4 = \underline{10\frac{2}{3}}$ units^2

Question 5 :

a) $\frac{dy}{dx} = 1 - 6\sin 3x \quad \Rightarrow \quad y = x + 2\cos 3x + c$

when $x = 0$, $y = 7$

$\therefore \quad 7 = 0 + 2\times 1 + c \quad \Rightarrow \quad c = 5$

Equation of curve is $\underline{y = x + 2\cos 3x + 5}$

b) (i) to have a limiting sum, $|r| < 1$

$\therefore |2x| < 1 \quad \Rightarrow \quad -1 < 2x < 1$

$\underline{-\frac{1}{2} < x < \frac{1}{2}}$

(ii) $S_\infty = \frac{5}{1-2x} = 100$

$\therefore \quad 1 = 20 - 40x \quad \Rightarrow \quad \underline{x = \frac{19}{40}}$

c) (i) $A = \underline{6000}$

(ii) $e^{-6k} = \frac{1}{6} \quad \Rightarrow \quad e^{6k} = 6$

$\therefore \quad \underline{k = \frac{1}{6}\ln 6}$

(iii) $\frac{dI}{ds} = -1000\ln 6 \times e^{\left(-\frac{\ln 6}{6}\right)s}$

when $s = 6$, $\frac{dI}{ds} = -1000\ln 6 \times e^{(-\ln 6)}$

$= \frac{-1000\ln 6}{6} \approx -298\cdot 6$

the light intensity is decreasing at a rate of 298·6 lux / m

Question 6 :

a) $\sin^2 \frac{x}{3} = \frac{1}{2} \quad \Rightarrow \quad \sin \frac{x}{3} = \frac{\pm 1}{\sqrt{2}}$

$\therefore \frac{x}{3} = \frac{\pi}{4}, \frac{-\pi}{4} \quad \Rightarrow \quad \underline{x = \frac{3\pi}{4}, \frac{-3\pi}{4}}$

b) (i) initial velocity is 20 m/s

(ii) when $\underline{t = 10}$

(iii) acceleration is zero at the maximum TP of $v = f(t)$

ie when $\underline{t = 6}$

(iv) distance travelled $= \int_0^8 v\, dt$

$d \approx \frac{2}{3}\left\{ 20 + 60 + 4\times(50+80) + 2\times 70 \right\}$

$= \underline{493\frac{1}{3}}$ metres

c) $V = \pi \int_3^6 \left(\frac{5}{x-2}\right)^2 dx$

$= \pi \int_3^6 25(x-2)^{-2} dx$

$= 25\pi \left[\frac{(x-2)^{-1}}{-1\times 1}\right]_3^6 = 25\pi\left(-\frac{1}{4} + 1\right)$

$= \frac{75\pi}{4}$ units^3

Question 7 :

a) let $u = \ln x$

$\therefore \quad u - \frac{3}{u} = 2 \quad \Rightarrow \quad u^2 - 2u - 3 = 0$

$$(u-3)(u+1) = 0$$

$\therefore \quad u = 3 \quad \Rightarrow \quad \underline{x = e^3}$

or $\quad u = -1 \quad \Rightarrow \quad \underline{x = e^{-1} = \frac{1}{e}}$

b) (i) since $\theta \le 2\pi$ then $r\theta \le 2\pi r$

and $l = r\theta = \frac{10\pi}{3} \quad \Rightarrow \quad \frac{10\cancel{\pi}}{3} \le 2\cancel{\pi} r$

$\therefore \quad \underline{r \ge \frac{5}{3}}$

(ii) when $r = 4$ in $l = r\theta$, $\quad \frac{10\pi}{3} = 4\theta \quad \Rightarrow \quad \theta = \frac{5\pi}{6}$

Now, using $A = \frac{1}{2}r^2\theta$, $\quad Area = \frac{1}{2} \times 4^2 \times \frac{5\pi}{6} = \frac{20\pi}{3}$ $\underline{\text{units}^2}$

c) (ii) $\quad P(G\ wins) = P(XGG) + P(GXG) + P(GG)$

$$= \frac{2}{3}\times\frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{2}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{3}$$

$$= \frac{7}{27}$$

(iii) P(*3 games played*) $= P(XG) + P(GX)$

$$= \frac{2}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{2}{3} = \frac{4}{9}$$

Question 8 :

a) (i) $y = x^4 - 8x^2$ crosses the coordinate axes at $\left(-2\sqrt{2}, 0\right)$ $\left(2\sqrt{2}, 0\right)$

and touches the axes at the point $(0,0)$

(ii) $f(-x) = (-x)^4 - 8(-x)^2$

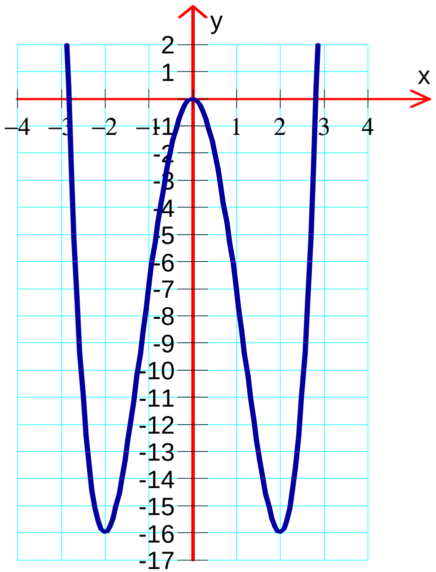
$= x^4 - 8x^2$

$= f(x)$

$\therefore \quad f(x)$ is an even function.

(iii) (0,0) is a maximum turning point
(-2, -16) is a minimum TP
(2, -16) is a minimum TP

(iv) see graph



b) (i) *CD* = *AD* (Opposite sides of a parallelogram equal)
AD = *EB* (*ABEF* is a square, all sides equal)
$\therefore$ *CD* = *BE*

(ii) In ΔBDC, ΔHEB

CD = *BE* (from part (i))
BC = *HB* (sides of square BCGH are equal)
$\angle BCD = \angle EBH$ ($\angle BCD = 180^0 - \angle ABC$ and
$\angle EBH = 360^0 - (180^0 + \angle ABC)$
$= 180^0 - \angle ABC$

$\therefore \quad \Delta BDC \equiv \Delta HEB$ (SAS)

$\therefore \quad BD = EH$ (corresponding sides of 2 congruent triangles equal)

Question 9 :

a) (i) $0.15 \times 0.15 = 0.0225 = \underline{2.25\%}$

(ii) $0.85 \times 0.2 = 0.17 = \underline{17\%}$

b) (i)
$$A_n = 100\,000 \times 1.005^n - M(1 + 1.005 + 1.005^2 + ... + 1.005^{n-1})$$
$$= 100\,000 \times 1.005^n - M\frac{\left(1.005^n - 1\right)}{1.005 - 1}$$
$$= 100\,000 \times 1.005^n - \frac{M\left(1.005^n - 1\right)}{0.005}$$

(ii) $A_{144} = 0 \quad \Rightarrow \quad 100\,000 \times 1.005^{144} = \dfrac{M\left(1.005^{144} - 1\right)}{0.005}$

$$\therefore \quad \underline{M = \$\,975.85}$$

c) (i) $f'(x) = kb^2 x - \dfrac{kx^3}{3} + c$

$f'(-b) = -f'(b) \quad \Rightarrow \quad c = 0 \quad , \quad \therefore \quad \underline{f'(x) = k\left(b^2 x - \dfrac{x^3}{3}\right)}$

(ii) $f(x) = \dfrac{kb^2x^2}{2} - \dfrac{kx^4}{12} + c_2$

since $f(b) = 0$ from graph,

$\dfrac{kb^2b^2}{2} - \dfrac{kb^4}{12} + c_2 = 0 \quad \Rightarrow \quad c_2 = \dfrac{-5kb^4}{12}$

$\therefore \quad f(x) = \dfrac{kb^2x^2}{2} - \dfrac{kx^4}{12} - \dfrac{5kb^4}{12} \quad \Rightarrow \quad f(0) = -\dfrac{5kb^4}{12}$

$\therefore$ The beam is $\dfrac{5kb^4}{12}$ units below the x axis.

Question 10 :

a) $y = \ln(x-2) \quad \Rightarrow \quad x = e^y + 2$

$$Area = 7 \times \ln 5 - \int_0^{\ln 5} \left(e^y + 2\right) dy = 7\ln 5 - \left[e^y + 2y\right]_0^{\ln 5}$$
$$= \underline{\left(5\ln 5 - 4\right)} \text{ unit}^2$$

b) (i) ΔKJO $|||$ ΔPMO (equiangular)

$\therefore \quad \dfrac{MP}{KJ} = \dfrac{OM}{OJ} \quad \Rightarrow \quad \dfrac{MP}{s} = \dfrac{l-x}{x} \quad \Rightarrow \quad MP = \frac{s}{x}(l-x)$

$A = \frac{1}{2}(sx)\sin\alpha + \frac{1}{2}(l-x)MP\sin\alpha$

$= \frac{1}{2}\sin\alpha\left\{xs + (l-x)\dfrac{s(l-x)}{x}\right\} = \underline{s(\sin\alpha)\left\{x - l + \dfrac{l^2}{2x}\right\}}$

(ii) $\dfrac{dA}{dx} = s(\sin\alpha)\left(1 - \dfrac{l^2}{2x^2}\right) = 0$, to minimise A.

$\therefore \quad 1 - \dfrac{l^2}{2x^2} = 0 \quad \Rightarrow \quad x = \dfrac{l}{\sqrt{2}}$

and $\dfrac{\partial^2 A}{\partial x^2} = s(\sin\alpha)\left(\dfrac{l^2}{x^3}\right) > 0$ for all $x > 0$.

$\therefore \quad x = \dfrac{l}{\sqrt{2}}$ gives a minimum value for A.

(iii) $MP = \dfrac{s}{\left(\frac{l}{\sqrt{2}}\right)}\left(l - \frac{l}{\sqrt{2}}\right) = \dfrac{s\cancel{\sqrt{2}}}{\cancel{l}}\left(\dfrac{\sqrt{2}-1}{\cancel{\sqrt{2}}}\right)\cancel{l}$

$\therefore \quad MP = \underline{s\left(\sqrt{2} - 1\right)}$