2 Unit 2004 HSC

Question 1

a) $3397000 = 3.4 \times 10^6$

b) $\frac{d}{dx}(x^4 + 5x^{-1})$

$= 4x^3 - 5x^{-2}$

c) $\frac{x-5}{3} - \frac{x+1}{4} = 5$

$\Rightarrow \frac{4x-20-3x-3}{12} = 5$

$\Rightarrow x = 60 + 23 = 83$

d) $(3-\sqrt{2})^2 = 9 - 2(3)\sqrt{2} + 2$

$= 11 - 6\sqrt{2}$

$\therefore a = 11 \ , \ b = 6$

e) $P(\text{red}) = 12/30$

$P(\text{yellow}) = 7/30$

$\therefore P(\text{red or yellow}) = \frac{12}{30} + \frac{7}{30}$

$= \frac{19}{30}$

f) $|x+1| \leq 5$

$-5 \leq x+1 \leq 5$

$-5-1 \leq x \leq 5-1$

$-6 \leq x \leq 4$

Question 2

i) $d_{AB} = \sqrt{(-1-2)^2 + (3-0)^2}$

$= \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

ii) $m_{AB} = \frac{3-0}{-1-2} = \frac{3}{-3} = -1$

iii) Since $m_{AB} = -1$

$m_{AB} = -1 = \tan\theta_{\angle xBD}$

$\theta_{\angle xBD} = 45°$

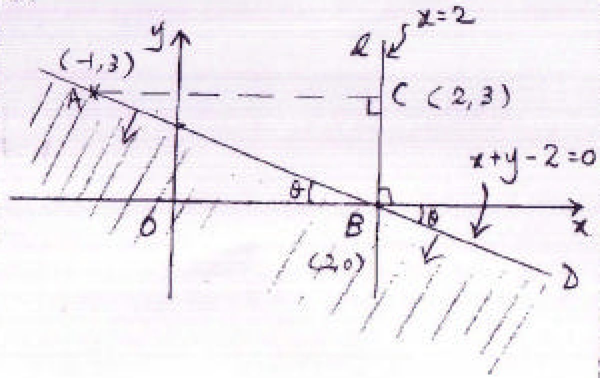
iv) Eqn of AB:

$\frac{y-0}{x-2} = -1$

$\Rightarrow y = -x + 2$

$\Rightarrow x + y - 2 = 0$

v)



vi) $x = 2$.

vii) $AC \perp AB$

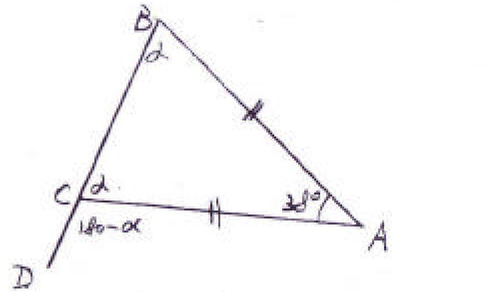
$\therefore$ C's y coordinate $= 3$

Since C is on the ℓ

$\therefore$ C's x coordinate $= 2$

$\therefore C = (2,3)$

b)



$\alpha = \frac{180 - 38°}{2} = 71°$ (base $\angle$ of isos Δ)

$\angle ACD = 180° - \alpha = 109°$

($\angle$ sum of Δ)

c) $x^2 - kx + 4 = 0$

For no real roots,

$b^2 - 4ac < 0$

$\Rightarrow (-k)^2 - 4(1)(4) < 0$

$\Rightarrow k^2 - 16 < 0$

$k^2 < 16$

$-4 < k < 4$

2004 2 Unit HSC

Question 3:

(a) (i) $\frac{d}{dx}(x^2 \log_e x)$

Let $u = x^2$, $u' = 2x$

$v = \log_e x$ $v' = \frac{1}{x}$.

$\therefore \frac{d}{dx}(x^2 \log_e x)$

$= u\frac{dv}{dx} + v\frac{du}{dx}$

$= x^2(\frac{1}{x}) + \log_e x\,(2x)$

$= x + 2x \log_e x$ *

ii) $\frac{d}{dx}(1+\sin x)^5$

Let $u = 1 + \sin x$

$u' = \cos x$

$\therefore \frac{d}{dx}(1+\sin x)^5$

$= 5(1+\sin x)^4 \cdot \cos x$.

$= 5\cos x(1+\sin x)^4$ *

1b) i) $\int_1^2 e^{3x}\,dx$

$= \left[\frac{1}{3}e^{3x}\right]_1^2$

$= \frac{1}{3}e^6 - \frac{1}{3}e^1$

$= \frac{1}{3}(e^6 - e)$

ii) $\int \frac{x}{x^2-3}\,dx$

Let $u = x^2 - 3$

$u' = 2x$

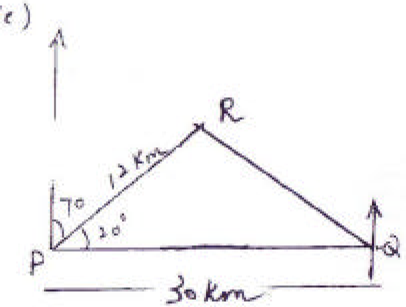
$\therefore \int \frac{x}{x^2-3}\,dx$

$= \frac{1}{2}\int \frac{2x}{x^2-3}\,dx$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\log(x^2-3) + c$ *

1c)



i) By Cosine Rule,

$a^2 = b^2 + c^2 - 2bc\cos A$

$RQ^2 = PR^2 + PQ^2 - 2(PR)(PQ)\cos 20°$

$= 12^2 + 30^2 - 2(12)(30)\cos 20°$

$= 1044 \quad - 676.58$

$= 367.42$

$RQ = 19.168203$

$= 19.17$ Km.

ii) By Sine Rule,

$\frac{PR}{\sin\angle PQR} = \frac{RQ}{\sin 20°}$

$\sin\angle PQR = \frac{\sin 20°}{RQ} * PR$

$= \frac{\sin 20°}{19.17} * 12$

$= 0.2140971$

$\angle PQR = 12.36°$

Bearing of R from Q = 12 + 270 = 282°T *

P2

Question 4

a) Length of Arc $= l = r\theta$

$r\theta = 5\pi$

$\theta = \frac{5}{6}\pi$

$\therefore$ Area of Sector AOB

$= \frac{1}{2} r^2 \theta$

$= \frac{1}{2}(6^2)(\frac{5}{6}\pi)$

$= 15\pi \text{ cm}^2$

b) $f(x) = x^3 - 3x^2$

i) $f'(x) = 3x^2 - 6x$

$= 3x(x-2)$

$f'(x) = 0$

$3x = 0$ or $x = 2$

when $x = 0$, $f(0) = 0$

$x = 2$, $f(2) = 8 - 3(4) = -4$

$f''(x) = 6x - 6$

$f''(0) = -6 < 0$

$\therefore (0,0)$ is the max.

$f''(2) = 12 - 6 = 6 > 0$

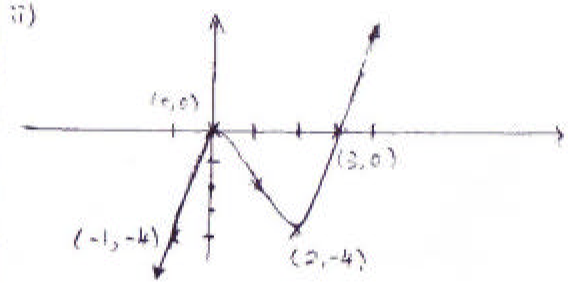
$\therefore (2,-4)$ is the min.

$f(x) = x^3 - 3x^2 = 0$

$\Rightarrow x^2(x-3) = 0$

$x = 0$ or $x = 3$

ii)



iii) $f(x)$ is concave up when $f''(x) > 0$

$6x - 6 > 0$

$x - 1 > 0$

$x > 1$

c) $V = \pi \int y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{3}} (2\sec x)^2 \, dx$

$= \pi \int_0^{\frac{\pi}{3}} 4\sec^2 x \, dx$

$= 4\pi \left[\tan x\right]_0^{\frac{\pi}{3}}$

$= 4\pi \left[\tan \frac{\pi}{3} - \tan 0\right]$

$= 4\pi \left[\sqrt{3} - 0\right]$

$= 4\sqrt{3}\pi \text{ units}^3$

Question 5

a) $a = 30$, $d = 5$

i) $T_{21} = a + (n-1)d$

$= 30 + (21-1)(5) = 30 + 100$

$= 130$

ii) $S_{21} = \frac{n}{2}(a + l) = \frac{21}{2}(30 + 130)$

$= 21(80) = 1680$ minutes

$= \frac{1680}{60} = 28$ hours

iii) $S_n = 50 \times 60 = 3000$ minutes

But $S_n = \frac{n}{2}(2a + (n-1)d)$

$\therefore 3000 = \frac{n}{2}(60 + 5n - 5)$

$6000 = n(55 + 5n)$

$5n^2 + 55n - 6000 = 0$

$n^2 + 11n - 1200 = 0$

$n = \frac{-11 \pm \sqrt{11^2 - 4(1)(-1200)}}{2(1)}$

$= \frac{-11 \pm \sqrt{70.15}}{2}$

$= \frac{59}{2}$ or $-\frac{81}{2}$ (rejected)

$= 29.5$

$= 30$th lesson

Question 5:

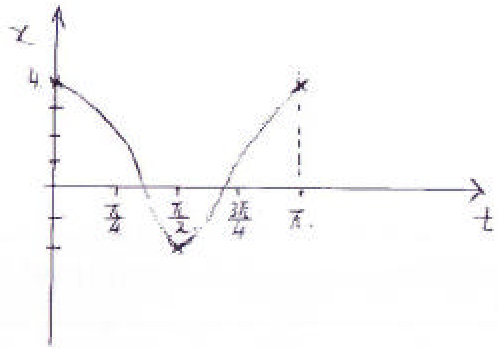
b) $x = 1 + 3\cos 2t$

i) Initial displacement when $t=0$

$\therefore$ At $t=0$, $x = 1 + 3\cos 0$

$= 1 + 3(1) = 4$ #

ii)



iii) Particle is stationary when $v=0$, ie at turning point

$\therefore t = \frac{\pi}{2}$

iv) $v = -6\sin 2t$

$-1 \le \sin 2t \le 1$

$\therefore$ max $|v|$ = speed = 6 m/s

this occurs when $\sin 2t = \pm 1$,

one value of $t = \frac{\pi}{4}$

Question 6:

a) $e^{2x} + 3e^{x} - 10 = 0$

$(e^{x} - 2)(e^{x} + 5) = 0$

$e^{x} = 2$ or $e^{x} = -5$ (rejected)

ie $e^{x} = \ln 2 \Rightarrow x = \ln 2$ #

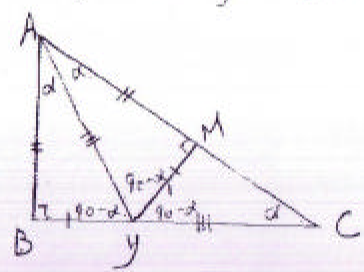
c) i) $AM = MC$ (given)

$\angle AMY = \angle CMY = 90^\circ$ ($Y \perp AC$ at M)

$MY = MY$ (common)

$\therefore \Delta AYM \equiv \Delta CYM$ (SAS)

ii)



$\angle BAY = \angle YAM = \alpha$ (given)

$\angle YCM = \angle YAM = \alpha$ ($\Delta AYM \equiv \Delta CYM$)

$\therefore \angle BAC + \angle ABC + \angle ACB = 180^\circ$

($\angle$ sum of Δ)

$\therefore \alpha = 30^\circ$

$\frac{YM}{YC} = \sin 30^\circ = \frac{1}{2}$

$\therefore$ By Pythagoras theorem

$MC = \sqrt{3}$.

$MY : AC = 1 : 2\sqrt{3}$ #

c)

i)

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

i) $P(\text{zero on 1st turn}) = \frac{11}{36}$ #

ii) $P(\geq 16) = \frac{4}{36} = \frac{1}{9}$ #

iii) $P(\geq 45) = \left[\frac{2}{36} \times \frac{1}{36} \times 2 + \frac{1}{36} \times \frac{1}{36}\right]$

$= \frac{4}{1296} + \frac{1}{1296} = \frac{5}{1296}$

$\therefore P(< 45) = 1 - \frac{5}{1296}$

$= \frac{1291}{1296}$ #

Question 7

(a) $\sum_{n=2}^{4} n^2 = 2^2 + 3^2 + 4^2 = 4 + 9 + 16 = 29$

(b) (i) $P = Ae^{kt}$

$\frac{dP}{dt} = Ake^{kt} = kP$

(ii) when $t=0$, $P = 17$ million $\therefore A = 17$ million

(iii) When $t = 13$, $P = 20$ million

$\therefore 20 = 17e^{13k} \Rightarrow k = \frac{1}{13}\ln\left(\frac{20}{17}\right)$

(iv) $30 = 17e^{kt}$

$\frac{30}{17} = e^{kt}$

$kt = \ln\left(\frac{30}{17}\right)$

$t = 45.43\ldots$

$\therefore$ Australia's population will reach 30 million in the year 2036.

c) $r = \frac{0.06}{12} = 0.005$

(i) $80\left(1 + \frac{r}{100}\right)^{300} = 80(1.005)^{300} = \$357.20$

(ii) $80(1.005)^{300} + 80(1.005)^{299} + \ldots + 80(1.005)^1$

Final value of Lois' trust $= 80(1.005)\left[1 + (1.005) + \ldots + (1.005)^{299}\right]$

$= 80(1.005)\left[\frac{(1.005)^{300} - 1}{0.005}\right]$

$= \$55,716.71$

Question 8

(a) (i) $\cos\theta\tan\theta = \cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) = \sin\theta$

(ii) $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\theta \quad 0 < \theta \leq 2\pi$

$8\sin\theta(\sin\theta) = \frac{1}{\sin\theta}$

$8\sin^2\theta = \frac{1}{\sin\theta}$

$\sin^3\theta = \frac{1}{8}$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

(b) (i) $y = -4$

(ii) $x^2 = 16y$

$y = \frac{x^2}{16}$

$\frac{dy}{dx} = \frac{2x}{16} = \frac{x}{8}$

When $x = 4$, $\frac{dy}{dx} = \frac{1}{2}$

$\therefore$ gradient of tangent at $A = \frac{1}{2}$

$(y - y_1) = m(x - x_1)$

$(y - 1) = \frac{1}{2}(x - 4)$

$2y - 2 = x - 4$

$x - 2y - 2 = 0 \quad \ldots\ldots$ (1)

(iii) Substitute $y = -4$ into (1):

$x - 2(-4) - 2 = 0 \Rightarrow x = -6$

(iv) $\int_{-8}^{4} \left(2 - \frac{x}{4}\right) - \frac{x^2}{16}\,dx$

$= \left[2x - \frac{x^2}{8} - \frac{x^3}{48}\right]_{-8}^{4}$

$= \left(8 - 2 - \frac{64}{48}\right) - \left(-16 - 8 + \frac{512}{48}\right)$

$= 18 \text{ units}^2$

(v) $C(-6, -4)$

$y = 2 - \frac{x}{4} \rightarrow 4y = 8 - x \rightarrow x + 4y - 8 =$

Need first to find the perpendicular distance from C to AB.

$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

$= \left|\frac{1 \times -6 + 4 \times -4 - 8}{\sqrt{1^2 + 4^2}}\right|$

$= \left|\frac{-6 - 16 - 8}{\sqrt{17}}\right| = \frac{30}{\sqrt{17}}$ units

$d_{AB} = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

$= \sqrt{3^2 + (-12)^2} = \sqrt{153} = 3\sqrt{17}$ units

Area of shaded Region $=$ Area ΔABC $-$ Area in (iv)

$= \frac{1}{2} \times 3\sqrt{17} \times \frac{30}{\sqrt{17}} - 18$

$= 45 - 18 = 27 \text{ units}^2$ P.S

Question 9

(a) (i) $r = -\tan^2\theta$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1+\tan^2\theta} = \frac{1}{\sec^2\theta} = \cos^2\theta$$

(ii) Limiting sum exists when $|r| < 1$

i.e. $\tan^2\theta < 1$ i.e. $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

(b) (i) $V_{max.}$ occurs when $a = 0$ $\therefore t = 2$.

(ii) Maximum displacement occurs when $v = 0$. This occurs when the areas above and below the t-axis are the same $\therefore t = 4$.

(c) (i) $f(x) = \frac{\log_e x}{x}$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \log_e x \cdot 1}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

let $f'(x) = 0$ to find s.p.

i.e. $1 - \log_e x = 0$

$\log_e x = 1$

$x = e$

(ii)

x	2.5	e	3
$f'(x)$	0.01	0	−0.01

$\therefore$ s.p. is a maximum.

(iii) when $x = e$, $f(e) = \frac{1}{e}$

$\therefore \frac{\log_e x}{x} \le \frac{1}{e}$

$e \log_e x \le x$

$\log_e x^e \le \log_e e^x$

$\therefore x^e \le e^x$ for all $x > 0$

Question 10

(a) (i) $\int_a^{3a} \frac{1}{x}\,dx \approx \frac{h}{3}\left[y_0 + y_n + 4y_1\right]$

$$= \frac{a}{3}\left[\frac{1}{a} + \frac{1}{3a} + 4\left(\frac{1}{2a}\right)\right]$$

$$= \frac{1}{3}\left[1 + \frac{1}{3} + 2\right]$$

$$= 1\tfrac{1}{9} = \frac{10}{9} \text{ units}^2$$

(ii) $\int_a^{3a} \frac{1}{x}\,dx = \left[\ln x\right]_a^{3a}$

$$= \ln 3a - \ln a = \ln\left(\frac{3a}{a}\right) = \ln 3$$

$\therefore \ln 3 \approx \frac{10}{9}$

(b) (i) Area of $\Delta ABC = 2\,\Delta AST$

i.e. $\frac{1}{2} bc \sin A = 2\left(\frac{1}{2} xy \sin A\right)$

$\frac{1}{2} bc \sin A = xy \sin A$

$xy = \frac{1}{2} bc$

(ii) $a^2 = b^2 + c^2 - 2bc \cos A$

$z^2 = x^2 + y^2 - 2xy \cos A$

$= x^2 + \left(\frac{bc}{2x}\right)^2 - bc\cos A$ $\left[\begin{matrix}\text{from (i)} \\ 2xy = bc\end{matrix}\right]$

$= x^2 + \frac{b^2c^2}{4x^2} - bc\cos A$

(iii) $\frac{dz^2}{dx} = 2x - \frac{b^2c^2}{2x^3} = 0$ to find max./min.

$2x = \frac{b^2c^2}{2x^3}$

$4x^4 = b^2c^2$

$x^4 = \frac{b^2c^2}{4}$

$x^2 = \pm \frac{bc}{2}$

$x = \pm\sqrt{\frac{bc}{2}}$, but $x > 0$

$\therefore x = \sqrt{\frac{bc}{2}}$

$\frac{d^2z^2}{dx^2} = 2 + \frac{3b^2c^2}{2x^4} > 0$ for $x = \sqrt{\frac{bc}{2}}$

Hence $x = \sqrt{\frac{bc}{2}}$ yields a minimum.

(iv) $x^2 = \frac{bc}{2}$, substituting into z^2 gives:

$$z^2 = \frac{bc}{2} + \frac{b^2c^2}{2bc} - bc\cos A$$

$$= \frac{b^2c^2 + b^2c^2 - 2b^2c^2\cos A}{2bc}$$

$$= \frac{\cancel{2}b^2c^2(1 - \cos A)}{\cancel{2bc}}$$

$= bc(1 - \cos A)$ $\left[\cos A = \frac{b^2 + c^2 - a^2}{2bc}\right]$

$$= bc\left[1 - \frac{b^2 + c^2 - a^2}{2bc}\right]$$

$$= \cancel{bc}\left[\frac{2bc - (b^2 + c^2 - a^2)}{2\cancel{bc}}\right]$$

$$= \frac{a^2 - b^2 - c^2 + 2bc}{2}$$

$$= \frac{(a + c - b)(a + b - c)}{2}$$

$$= \frac{(P - 2b)(P - 2c)}{2}$$

$\therefore z = \sqrt{\frac{(P-2b)(P-2c)}{2}}$ m