2002 HSC MATHEMATICS EXAM – SOLUTIONS

Question 1

(a)

$$\frac{5.8^2 - 3.1^3}{3 * 3.1 * 5.8}$$

$$= \frac{33.64 - 29.791}{53.94}$$

$$= 0.0714$$

(to three significant figures)

(b)

$$d / dx(x^3 + 2)$$

$$= 3x^2$$

(c)

$$x^2 = 5x$$

$$x^2 - 5x = 0$$

$$x(x-5) = 0$$

$$x = 0, x = 5$$

(d)

$$\int \frac{3}{x} dx$$

$$= 3\ln x + C$$

(e)

$$3x - \frac{2x-5}{2} = 6$$

$$6x - (2x - 5) = 12$$

$$4x + 5 = 12$$

$$4x = 7$$

$$x = \frac{7}{4} = 1.75$$

(f)

$$x - 2y = 8$$

$$2x + y = 1$$

$$x - 2(1 - 2x) = 8$$

$$x - 2 + 4x = 8$$

$$5x = 10$$

$$x = 2$$

$$2(2) + y = 1$$

$$y = -3$$

Question 2

(a)

$y = e^{2x}$

$y' = 2e^{2x}$

$m = 2e^{2(0)} = 2$

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 0)$

$y = 2x + 1$

(b)

(i)

$d/dx(x.\sin x)$

$u = x \qquad v = \sin x$

$u' = 1 \qquad v' = \cos x$

$vu' + uv'$

$= \sin x + x.\cos x$

(ii)

$d/dx\left(\frac{\ln x}{x^2}\right)$

$u = \ln x \qquad v = x^2$

$u' = \frac{1}{x} \qquad v' = 2x$

$\frac{vu' - uv'}{v^2}$

$= \frac{x - 2x.\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$

(c)

$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$

$\frac{\sin 60}{x} = \frac{\sin 45}{y}$

$\frac{x}{y} = \frac{\sin 60}{\sin 45}$

$\frac{x}{y} = \frac{\sqrt{3}\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$

(d)

(i)

$\int (\cos 3x)dx$

$= \frac{1}{3}\sin 3x + C$

(ii)

$\int_0^1 (e^{5x} - 1)dx$

$= \left[\frac{1}{5}e^{5x} - x\right]_0^1$

$= \left[\left(\frac{1}{5}e^5 - 1\right) - \left(\frac{1}{5}\right)\right]$

$= \frac{e^5 - 6}{5}$

Question 3

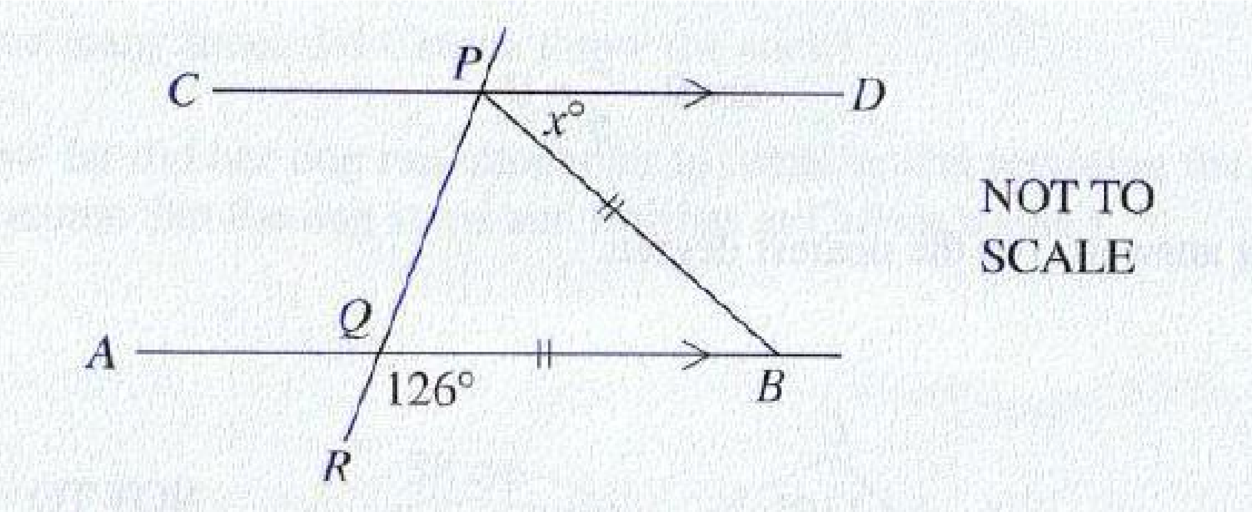
(a)

$A = P(1+r)^n$

$A = 1000(1.035)^{20}$

$A = \$1989.79$

(b)



$< BQP = 180 - 126$ (angles on a straight line add to 180 degrees)

$< BQP = 54$

$< BPQ =< BQP$ (base angles of isosceles triangle BPQ are equal)

$< BPQ = 54$

$< DPQ =< BQR$ (corresponding angles are equal)

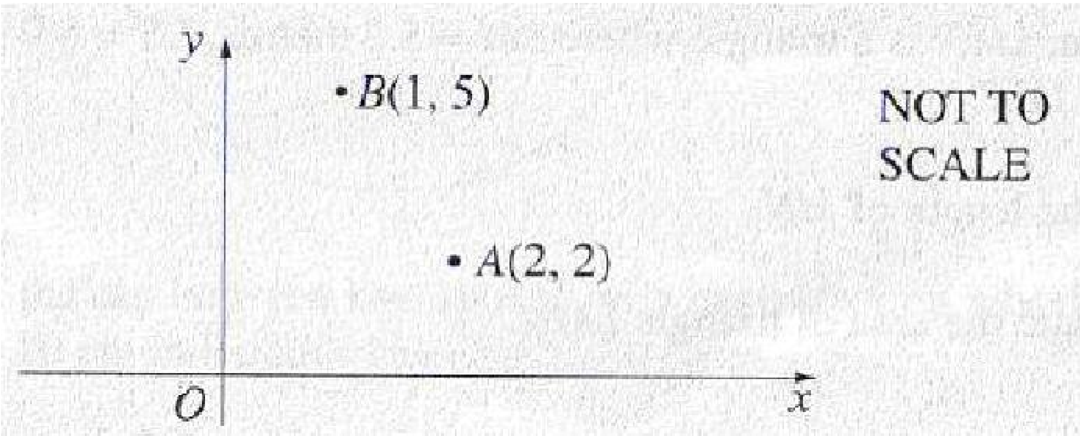
$< DPQ = 126$

$< DPQ =< BPQ + x$

$126 = 54 + x$

$x = 72$

(c)



(i)

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$M = \left(\frac{1 + 2}{2}, \frac{5 + 2}{2} \right)$$

$$M = \left(\frac{3}{2}, \frac{7}{2} \right)$$

(ii)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = -3$$

$$m_1 m_2 = -1$$ (for perpendicular lines)

$$\therefore m_2 = \frac{1}{3}$$

$$y - y_1 = m_2(x - x_1)$$ (using the midpoint)

$$y - \frac{7}{2} = \frac{1}{3}\left(x - \frac{3}{2} \right)$$

$$x - 3y + 9 = 0$$

(iii)
Point C lies on the y-axis, and so has an x-coordinate of 0.

Method 1

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\sqrt{(-1)^2 + (y_2 - 5)^2} = \sqrt{(-2)^2 + (y_2 - 2)^2}$$

$$1 + y_2^2 - 10y_2 + 25 = 4 + y_2^2 - 4y_2 + 4$$

$$26 = 8 + 6y_2$$

$$y_2 = 3$$

Method 2

$$x - 3y + 9 = 0$$

$$0 - 3y + 9 = 0$$

$$\therefore y = 3$$

As this line bisects AB, AC must equal BC.

And so the coordinates of point C are (0, 3).

(iv)

$$D(x,5)$$

$$x - 3y + 9 = 0$$

$$x - 3(5) + 9 = 0$$

$$x = 6$$

$$D(6,5)$$

(v)

$$BD = 6 - 1 = 5$$

$$h = 5 - 2 = 3$$

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}5 * 3$$

$$A = 7.5 units^2$$

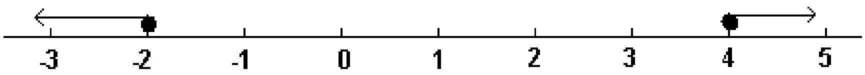
Question 4

(a)

$|x-1| \geq 3$ $\quad$ $|x-1| \geq 3$

$x-1 \geq 3$ $\quad$ $x-1 \leq -3$

$x \geq 4$ $\quad$ $x \leq -2$



(b)

$\cos q - \frac{2}{5} = 0$

$\cos q = \frac{2}{5}$

$q = \cos^{-1}\frac{2}{5}$

$q = 66^o, 294^o \ for 0 \leq q \leq 360$

(c)

(i)

$a^2 = b^2 + c^2 - 2bc.\cos A$

$MN^2 = 5.2^2 + 8.9^2 - 2(5.2)(8.9)\cos(110)$

$MN^2 = 137.9073845...$

$MN = 11.74$

(two decimal places)

(ii)

$A = \frac{1}{2}ab.\sin C$

$A = \frac{1}{2}(5.2)(8.9).\sin(110)$

$A = 21.74 units^2$

(d)

(i)

$6x - x^2 = 2x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, x = 4$

$y = 2x$

$y = 2(4)$

$y = 8$

$\therefore B(4,8)$

(ii)

$\int_0^4 (6x - x^2)dx - \int_0^4 2x dx$

$= \int_0^4 (6x - x^2 - 2x)dx$

$= \int_0^4 (4x - x^2)dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \left(2(4)^2 - \frac{4^3}{3}\right) - 0$

$= \frac{32}{3} units^2$

Question 5

(a)

(i)

$$T_n = a + (n-1)d$$
$$32 = 2 + (n-1)1.5$$
$$n = 21$$

(ii)

$$S_n = \frac{n}{2}(a+l)$$
$$S_{21} = \frac{21}{2}(2+32)$$
$$S_{21} = 357$$

$$2 * 357$$
$$= 714m$$

(b)

$$l = rq$$
$$38 = 20.q$$
$$q = 1.9rads$$

$$1.9 * \frac{180}{p}$$
$$= 109^o$$

(to the nearest degree)

(c)

(i)

$$y = x^2 - 8x + 4$$
$$y = x^2 - 8x + 4 + 12 - 12$$
$$y = x^2 - 8x + 16 - 12$$
$$(x-4)^2 = y + 12$$

$$\therefore Vertex : (4, -12)$$

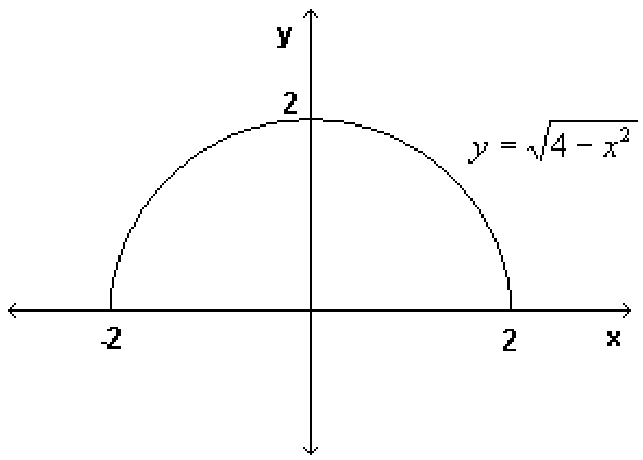
(ii)

$$(x-4)^2 = y + 12$$
$$(x-4)^2 = 4a(y+12)$$
$$(x-h)^2 = 4a(y-k)$$
$$\therefore a = \frac{1}{4}$$
$$Focus : (h, a+k)$$
$$= \left(4, \frac{1}{4} - 12\right)$$
$$= \left(4, -\frac{47}{4}\right)$$

Question 6

(a)

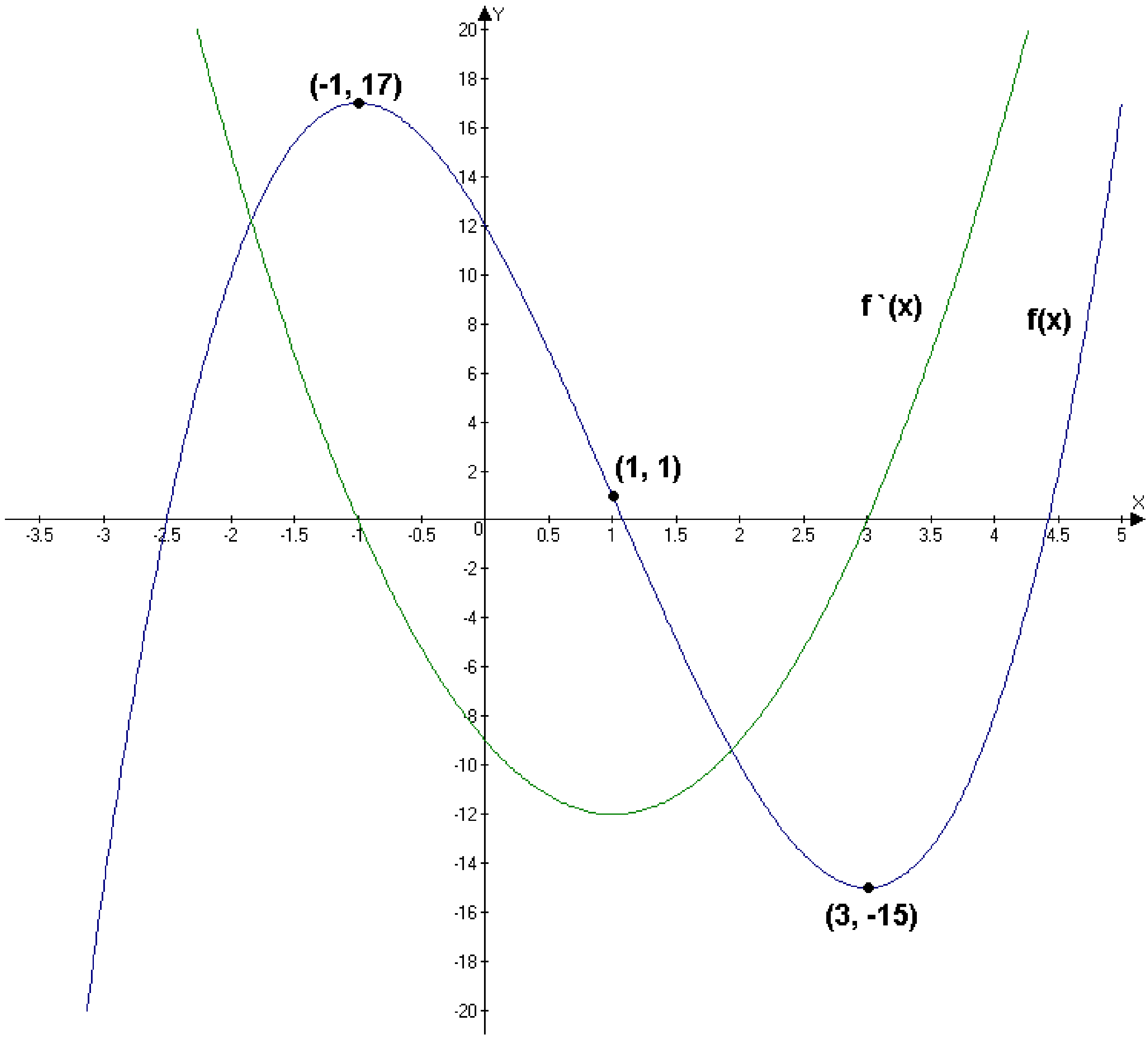
$Range : 0 \le y \le 2$



(b) (i)

$$f'(x) = 3(x+1)(x-3)$$
$$f'(x) = 3x^2 - 6x - 9$$
$$f(x) = x^3 - 3x^2 - 9x + c$$
$$when : x = 0,\ f(x) = 12$$
$$\therefore c = 12$$
$$f(x) = x^3 - 3x^2 - 9x + 12$$

(ii)



(c)

$$V = p\int_a^b x^2\,dy$$

$$y = \frac{x^4}{4}$$

$$x^4 = 4y$$

$$x^2 = 2\sqrt{y}$$

$$V = p\int_0^4 \left(2\sqrt{y}\right)dy$$

$$V = p\int_0^4 \left(2y^{\frac{1}{2}}\right)dy$$

$$V = p\left[\frac{4y^{\frac{3}{2}}}{3}\right]_0^4$$

$$V = p\left[\frac{4(4)^{\frac{3}{2}}}{3} - 0\right]$$

$$V = \frac{32p}{3}\,units^3$$

Question 7

(a)

(i)

$$r = \frac{T_3}{T_2} = \frac{T_2}{T_1}$$

$$r = (\sqrt{5} - 2)$$

$$r \approx 0.236$$

$$\therefore -1 < r < 1$$

As the common ratio is between –1 and 1, increasingly smaller values are being added to the series, and so it has a limiting sum.

(ii)

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{1}{1-(\sqrt{5}-2)}$$

$$S_\infty = \frac{1}{3-\sqrt{5}} * \frac{3+\sqrt{5}}{3+\sqrt{5}}$$

$$S_\infty = \frac{3+\sqrt{5}}{4}$$

(b)

(i)

$$V = 25\left(1 - \frac{t}{60}\right)^2$$

$$V_0 = 25\left(1 - \frac{0}{60}\right)^2$$

$$V_0 = 25L$$

(ii)

$$\frac{25}{4} = 25\left(1 - \frac{t}{60}\right)^2$$

$$\frac{1}{2} = 1 - \frac{t}{60}$$

$$t = 30s$$

(iii)

$$V = 25\left(1 - \frac{t}{60}\right)^2$$

$$\frac{dV}{dt} = -\frac{5}{6}\left(1 - \frac{t}{60}\right)$$

$$when: t = 30$$

$$\frac{dV}{dt} = -\frac{5}{12} L/s$$

(c) (i)

The sock that Chris picks first is irrelevant. Once he has picked the first sock, there are 7 socks left in the drawer, 1 of which matches his first sock. In order for him to not have a pair after choosing the second sock, he must choose one of the 6 socks that do not match his first sock. The probability of this occurring is $\frac{6}{7}$.

(ii)

$$\frac{6}{7} * \frac{4}{6} = \frac{4}{7}$$

(iii)

$$P(MM\,?) + P(M\,?\,M) + P(?\,MM)$$

$$= \frac{1}{7} + \left(\frac{6}{7} * \frac{1}{6}\right) + \left(\frac{6}{7} * \frac{1}{6}\right)$$

$$= \frac{3}{7}$$

Question 8

(a)(i)

$$Q = Q_0 e^{-kt}$$

$$6 = Q_0 e^{-k(0)}$$

$$Q_0 = 6$$

$$\therefore Q = 6e^{-kt}$$

$$\frac{6}{2} = 6e^{-k(15)}$$

$$\frac{1}{2} = e^{-15k}$$

$$-15k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln 2}{15} = 0.0462...$$

$$\therefore Q = 6e^{-\frac{\ln 2}{15}t}$$

(ii)

$$\frac{6}{8} = 6e^{-\frac{\ln 2}{15}t}$$

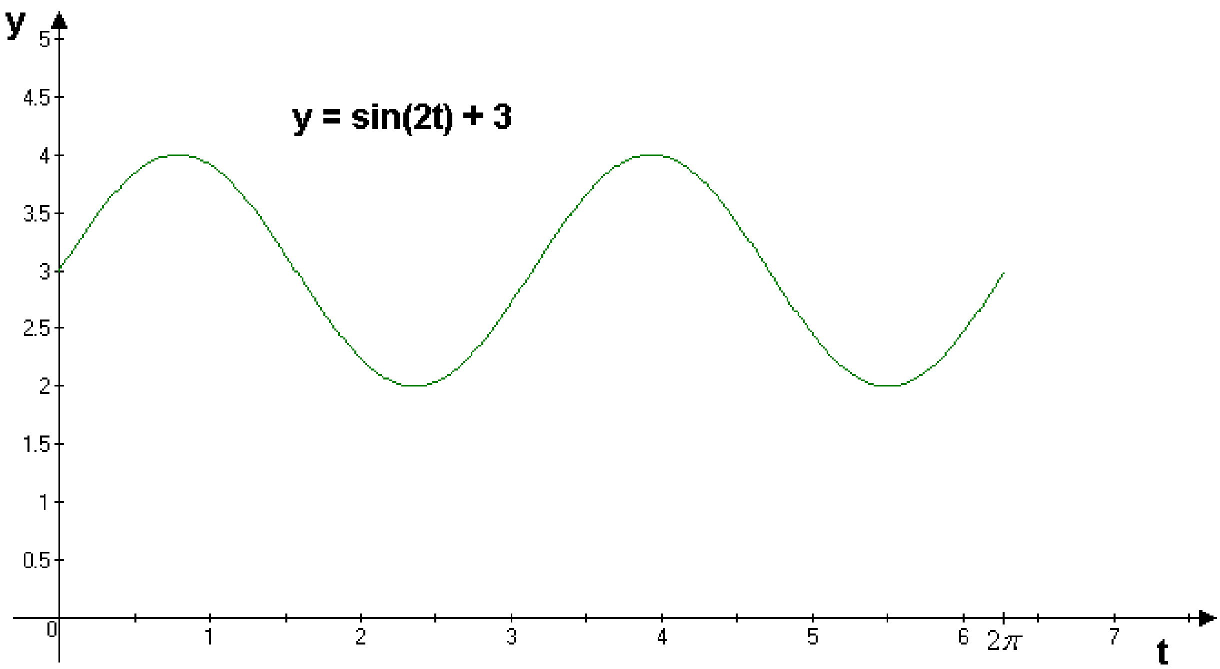
$$\frac{1}{8} = e^{-\frac{\ln 2}{15}t}$$

$$-\frac{\ln 2}{15}t = \ln\frac{1}{8}$$

$$t = \frac{15\ln 8}{\ln 2} = \frac{15\ln(2^3)}{\ln 2}$$

$$t = 45h$$

(b)(i)



(ii)

The particle is at rest when it has a velocity of zero. Velocity is zero when displacement is a maximum or a minimum.

Therefore, the particle will be at rest when $t = \frac{p}{4}, \frac{3p}{4}, \frac{5p}{4}, \frac{7p}{4}$.

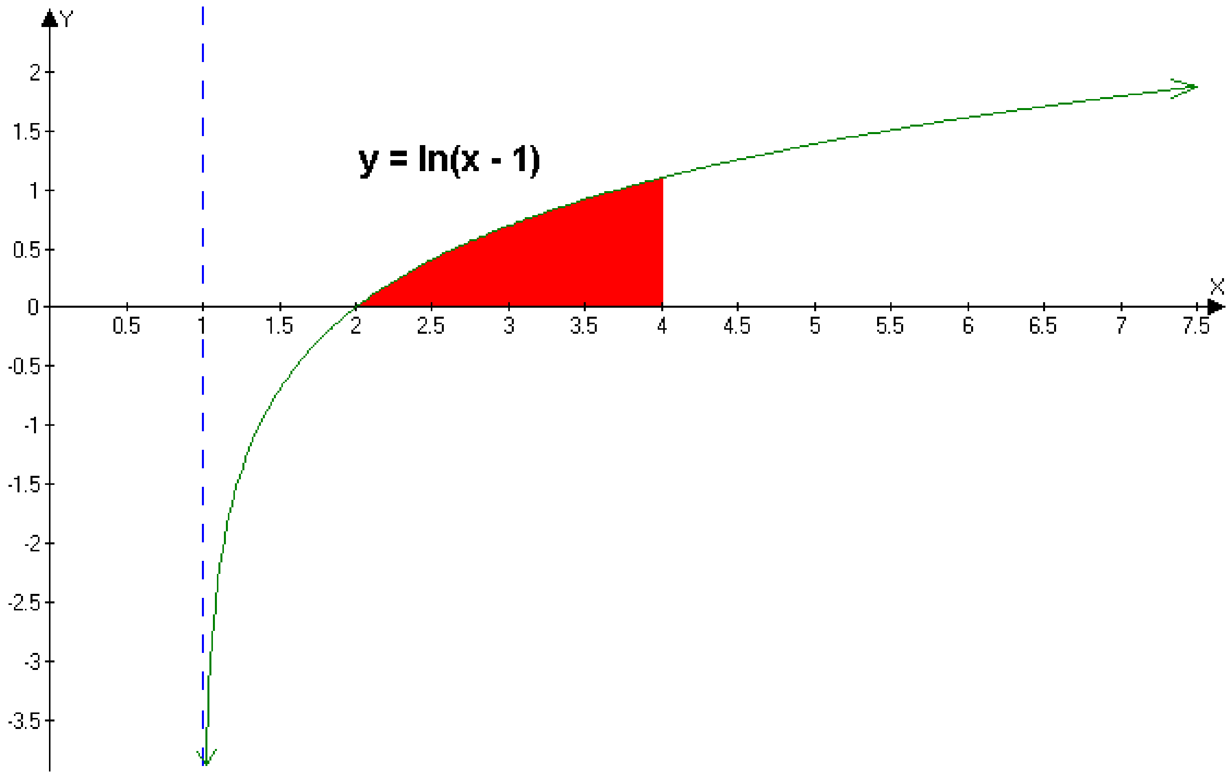
The particle will alternate between being 4 metres and 2 metres away from the origin.

(iii)

Initially, the particle begins at a distance of 3 metres from O and is already moving away from it at a decreasing rate. After $\frac{p}{4}$ seconds, it has stopped completely, only having travelled 1 metre. The particle then begins to accelerate towards O. At $t = \frac{p}{2}$ seconds, the particle is at a distance of 3 metres from O and it begins to decelerate. The particle comes to a stop just 2 metres away from O, when $t = \frac{3p}{4}$. The particle then accelerates again, away from the origin, and passes through its starting point at $t = p$ seconds. This oscillating motion is repeated indefinitely.

Question 9

(a)
(i)



(ii)

x	2	3	4
y	0	ln(2)	ln(3)
	y_0	y_1	y_2

$$\int_2^4 \ln(x-1)dx \approx \frac{h}{3}\left(y_0 + 4(y_1 + y_3 + y_5 \ldots) + 2(y_2 + y_4 + y_6 \ldots) + y_n\right)$$

$$= \frac{1}{3}\left(0 + 4(\ln 2) + \ln 3\right)$$

$$= \frac{4\ln 2 + \ln 3}{3}$$

$$= 1.29$$

(b)

$A_1 = 5000(1.0875)$

$A_2 = A_1(1.0875) + 5000(1.0875)$
$A_2 = 5000(1.0875)^2 + 5000(1.0875)$

$A_3 = A_2(1.0875) + 5000(1.0875)$
$A_3 = 5000(1.0875)^3 + 5000(1.0875)^2 + 5000(1.0875)$
$A_3 = 5000(1.0875 + 1.0875^2 + 1.0875^3)$

$A_n = 5000(1.0875 + 1.0875^2 + 1.0875^3 + ... + 1.0875^n)$

$A_{21} = 5000(1.0875 + 1.0875^2 + 1.0875^3 + ... + 1.0875^{21})$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{21} = \frac{1.0875(1.0875^{21} - 1)}{0.0875}$$
$S_{21} \approx 59.92$

$A_{21} = 5000 * S_{21}$
$A_{21} = \$299,604.86$

(c)
(i)

$v_1 = ct$

$v_2 = 2t^2$

$when: t = 5, v = 50$

$ct = 2t^2$

$c(5) = 2(5)^2$

$c = 10$

$\therefore v_1 = 10t$

(ii)

$d_1 = \int_0^5 (10t)dt$

$d_1 = \left[5t^2\right]_0^5$

$d_1 = 125m$

$d_2 = \int_0^5 (2t^2)dt$

$d_2 = \left[\frac{2t^3}{3}\right]_0^5$

$d_2 = 83.\dot{3}m$

$d_1 - d_2 = 125 - 83.\dot{3}$

$= 41.\dot{6}m$

After 5 seconds, the jet is approximately 42 metres behind the car.

(iii)

$d_1 = d_2$

$\int_0^t (10t)dt = \int_0^t (2t^2)dt$

$\left[5t^2\right]_0^t = \left[\frac{2t^3}{3}\right]_0^t$

$5t^2 = \frac{2t^3}{3}$

$0 = 2t^3 - 15t^2$

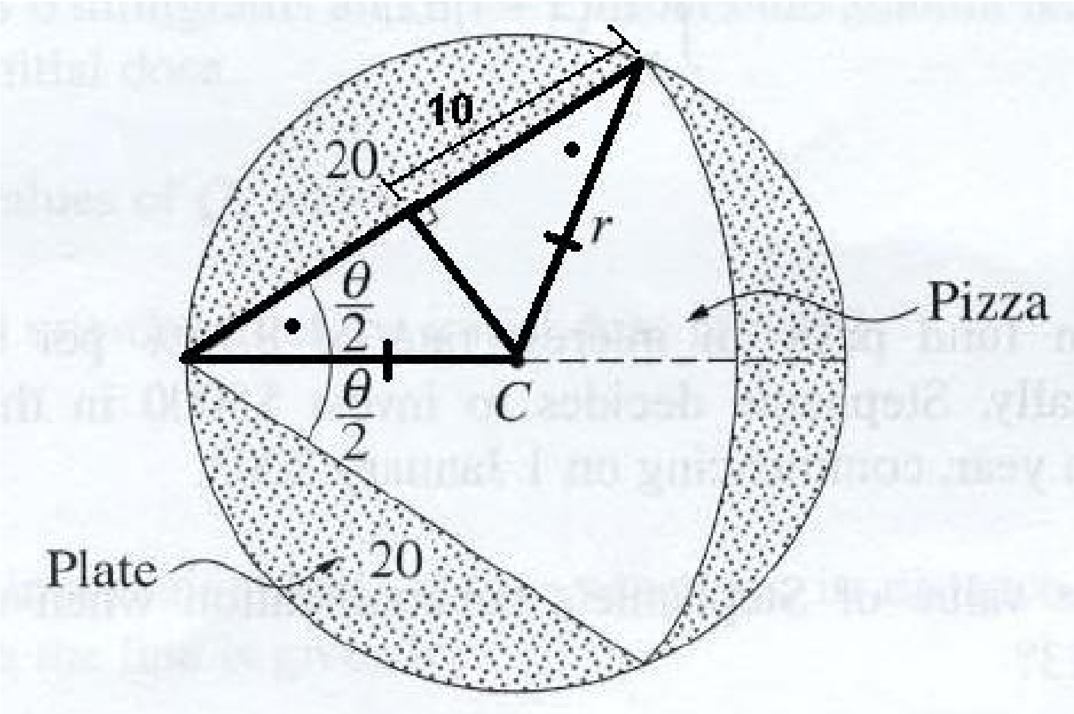
$0 = t^2(2t - 15)$

$t = 0, t = \frac{15}{2}$

The jet will have caught up with the car after 7.5 seconds have passed.

Question 10

(a)
(i)

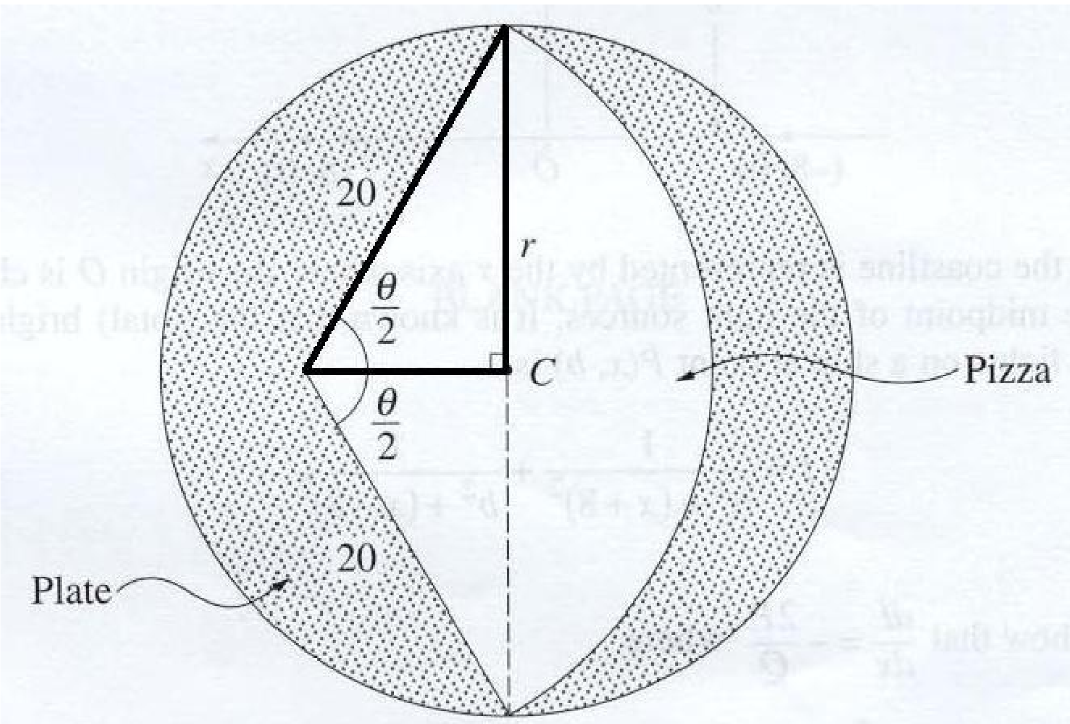


$$\cos\left(\frac{J}{2}\right) = \frac{10}{r}$$

$$r = \frac{10}{\cos\left(\frac{J}{2}\right)}$$

$$r = 10\sec\left(\frac{J}{2}\right)$$

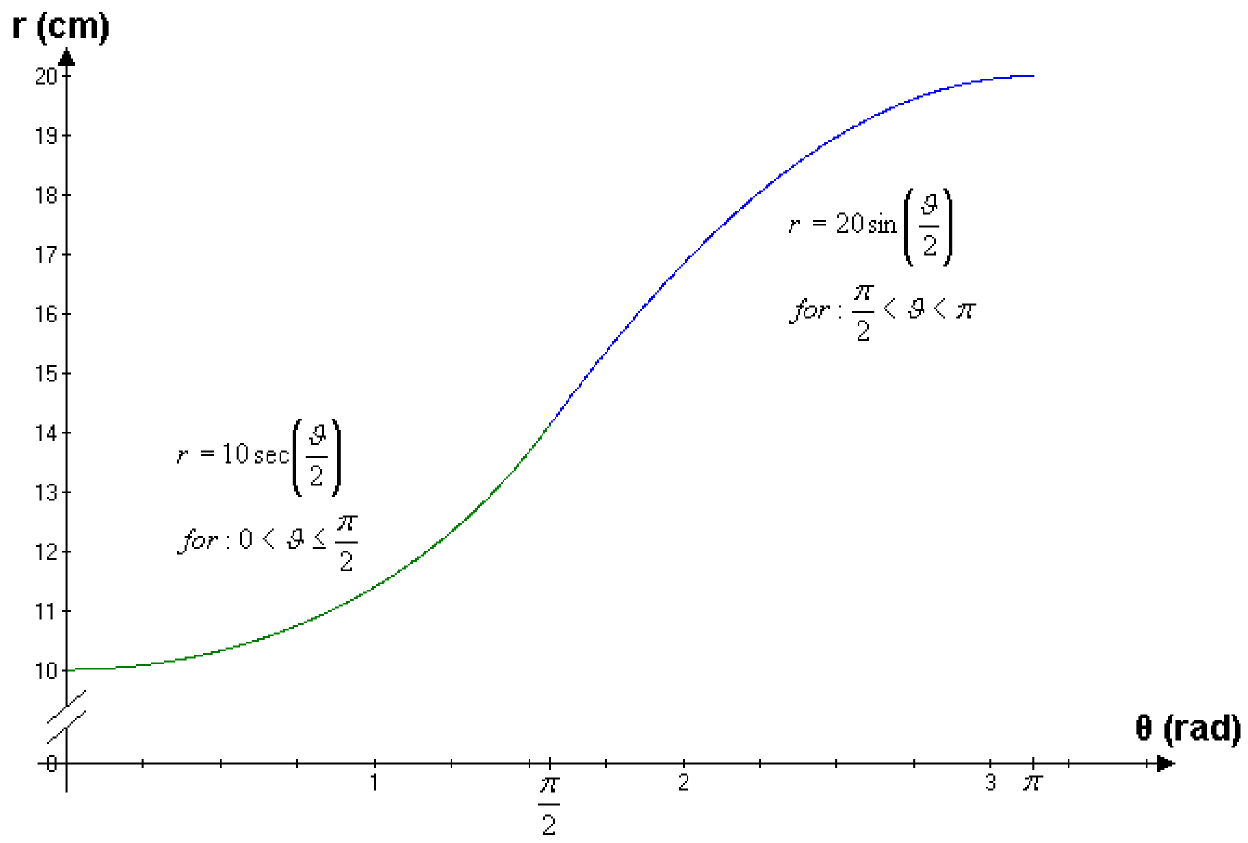
(ii)



$$\sin\left(\frac{J}{2}\right) = \frac{r}{20}$$

$$r = 20\sin\left(\frac{J}{2}\right)$$

(iii)



(b)

(i)

$$I = \frac{1}{b^2 + (x+8)^2} + \frac{1}{b^2 + (x-8)^2}$$
$$I = (b^2 + (x+8)^2)^{-1} + (b^2 + (x-8)^2)^{-1}$$

$$\frac{dI}{dx} = -2(x+8)(b^2 + (x+8)^2)^{-2} - 2(x-8)(b^2 + (x-8)^2)^{-2}$$
$$\frac{dI}{dx} = \frac{-2(x+8)}{(b^2 + (x+8)^2)^2} - \frac{2(x-8)}{(b^2 + (x-8)^2)^2}$$
$$\frac{dI}{dx} = \frac{-2(x+8)(b^2 + (x-8)^2)^2 - 2(x-8)(b^2 + (x+8)^2)^2}{(b^2 + (x+8)^2)^2(b^2 + (x-8)^2)^2}$$
$$\frac{dI}{dx} = \frac{-2\left[(x+8)(b^2 + (x-8)^2)^2 + (x-8)(b^2 + (x+8)^2)^2\right]}{(b^2 + (x+8)^2)^2(b^2 + (x-8)^2)^2}$$

$$\therefore \frac{dI}{dx} = -\frac{2P}{Q}$$

(ii)

Saga is 15 km from the coast, therefore $b = 15$.

$$\frac{dI}{dx} = -\frac{2P}{Q}$$

$$\frac{dI}{dx} = -\frac{4x\left(x^2 + 64 + b^2 + 16\sqrt{64 + b^2}\right)\left(x^2 + 64 + b^2 - 16\sqrt{64 + b^2}\right)}{(b^2 + (x+8)^2)^2 (b^2 + (x-8)^2)^2}$$

$$But : b = 15$$

$$\frac{dI}{dx} = -\frac{4x\left(x^2 + 561\right)\left(x^2 + 17\right)}{(225 + (x+8)^2)^2 (225 + (x-8)^2)^2}$$

$$Let : x = 0$$

$$Then : 4x = 0$$

$$\therefore \frac{dI}{dx} = 0$$

Therefore a stationary point exists at $x = 0$.

$$Test : x = 1$$

$$\frac{dI}{dx} = -\frac{4(1)\left((1)^2 + 561\right)\left((1)^2 + 17\right)}{(225 + ((1)+8)^2)^2 (225 + ((1)-8)^2)^2}$$

$$\frac{dI}{dx} = -\frac{4(562)(18)}{(225 + 81)^2 (225 + 49)^2}$$

$$\frac{dI}{dx} < 0$$

$\therefore$ I is decreasing.

$$Test : x = -1$$

$$\frac{dI}{dx} = -\frac{4(-1)\left((-1)^2 + 561\right)\left((-1)^2 + 17\right)}{(225 + ((-1)+8)^2)^2 (225 + ((-1)-8)^2)^2}$$

$$\frac{dI}{dx} = \frac{4(562)(18)}{(225 + 49)^2 (225 + 81)^2}$$

$$\frac{dI}{dx} > 0$$

$\therefore$ I is increasing.

x	-1	0	1
$\frac{dI}{dx}$	/	–	\

$\therefore x = 0$ is a maximum.

(iii)

Hero is 6 km from the coast, therefore $b = 6$.

$$\frac{dI}{dx} = -\frac{4x\left(x^2+64+b^2+16\sqrt{64+b^2}\right)\left(x^2+64+b^2-16\sqrt{64+b^2}\right)}{(b^2+(x+8)^2)^2(b^2+(x-8)^2)^2}$$

$$But: b = 6$$

$$\frac{dI}{dx} = -\frac{4x\left(x^2+260\right)\left(x^2-60\right)}{(36+(x+8)^2)^2(36+(x-8)^2)^2}$$

$$Let: \frac{dI}{dx} = 0$$

$$0 = -\frac{4x\left(x^2+260\right)\left(x^2-60\right)}{(36+(x+8)^2)^2(36+(x-8)^2)^2}$$

$$0 = -4x$$
$$\therefore x = 0$$

$$0 = x^2+260$$
$$x^2 = -260$$
$$x = \pm\sqrt{-260}$$

$\therefore$ no solution

$$0 = x^2-60$$
$$x^2 = 60$$
$$x = \pm 2\sqrt{15}$$

$$Test: x = 8$$

$$\frac{dI}{dx} = -\frac{4(8)\left((8)^2+260\right)\left((8)^2-60\right)}{\left(36+((8)+8)^2\right)^2\left(36+((8)-8)^2\right)^2}$$

$$\frac{dI}{dx} = -\frac{32(324)(4)}{\left(36+256\right)^2\left(36\right)^2}$$

$$\frac{dI}{dx} < 0$$

$\therefore$ I is decreasing.

$$Test: x = 1$$

$$\frac{dI}{dx} = -\frac{4(1)\left((1)^2+260\right)\left((1)^2-60\right)}{(36+((1)+8)^2)^2(36+((1)-8)^2)^2}$$

$$\frac{dI}{dx} = -\frac{4(261)(-59)}{(36+81)^2(36+49)^2}$$

$$\frac{dI}{dx} > 0$$

$\therefore$ I is increasing.

$Test: x = -1$

$$\frac{dI}{dx} = -\frac{4(-1)\left((-1)^2 + 260\right)\left((-1)^2 - 60\right)}{(36 + ((-1)+8)^2)^2 (36 + ((-1)-8)^2)^2}$$

$$\frac{dI}{dx} = \frac{4(261)(-59)}{(36+49)^2(36+81)^2}$$

$$\frac{dI}{dx} < 0$$

$\therefore$ I is decreasing.

$Test: x = -8$

$$\frac{dI}{dx} = -\frac{4(-8)\left((-8)^2 + 260\right)\left((-8)^2 - 60\right)}{(36 + ((-8)+8)^2)^2 (36 + ((-8)-8)^2)^2}$$

$$\frac{dI}{dx} = \frac{32(324)(4)}{(36)^2(36+256)^2}$$

$$\frac{dI}{dx} > 0$$

$\therefore$ I is increasing.

x	-8	$-2\sqrt{15}$	-1	0	1	$2\sqrt{15}$	8
$\frac{dI}{dx}$	/	—	\	—	/	—	\

As Hero sails from left to right, the total brightness from the lights increases until it reaches a maximum at a point ≈ 7.75 km to the left of the origin. The brightness then decreases and becomes a minimum when Hero is directly opposite the origin. As Hero continues to sail to the right, the brightness increases once again, becoming a second maximum at a point ≈ 7.75 km to the right of the origin. Beyond this point, the brightness on Hero gradually decreases.

NB: There are many different ways to do this question.