HSC 2003 – 2 Unit Solutions

Question 1

(a) $e^{-3.5} = 0.030197383 = 0.0302$ to 3 s.f.

(b) $\frac{d}{dx}(3x + \tan x) = 3 + \sec^2 x$

(c) $r = l\theta_{rad}$.

$3 = 5\theta_{rad}$. i.e. $\theta_{rad} = \frac{3}{5} = 0.6$

$\therefore \theta = \frac{0.6}{\pi} \times 180 = 34°$ to nearest degree.

(d) $112.5\% = \$315$

$1\% = \$2.80$

$100\% = \$280$

$\therefore$ cost of meal without tip $= \$280$

(e) $\int 3x^2 - 8\, dx = x^3 - 8x + c$

(f) $|x-3| = 7$

$x - 3 = 7$ OR $x - 3 = -7$

$x = 10$ $x = -4$

Question 2

(a) $y = 2\log_e x \rightarrow y' = \frac{2}{x}$

at $x = e$, $y' = \frac{2}{e}$ $\therefore$ grad. of normal $= -\frac{e}{2}$

Thus equation of normal is given by:

$(y-2) = -\frac{e}{2}(x-e)$

$2y - 4 = -ex + e^2$

$2y = -ex + e^2 - 4$

(b) (i) $d = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$

$= \sqrt{1^2 + (-1)^2} = \sqrt{2}$ units

(ii) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{-1-0} = \frac{1}{-1} = -1$

(iii) $m = \tan\theta$ $\therefore \theta = \tan^{-1}(m)$

$= \tan^{-1}(-1) = 135°$

(iv) Gradient of BC $= -1$ (OA ∥ BC)

$\therefore$ equation of BC is given by:

$(y-6) = -1(x-4)$

$y - 6 = -x + 4$

$y = -x + 10$

At $y = 0$, $x = 10$ $\therefore$ C(10, 0)

(v) $d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$, $x + y - 10 = 0$, O(0,0)

$= \left|\frac{1\times 0 + 1\times 0 - 10}{\sqrt{1^2+1^2}}\right|$

$= \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$ units

(vi) $d_{BC} = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$

$= \sqrt{(6-0)^2 + (4-10)^2}$

$= \sqrt{6^2 + (-6)^2} = \sqrt{72} = 6\sqrt{2}$ units

Area of OABC $= \frac{1}{2}(a+b)h$

$= \frac{1}{2}(\sqrt{2} + 6\sqrt{2}) . 5\sqrt{2}$

$= \frac{1}{2}(7\sqrt{2}) . 5\sqrt{2}$

Question 3

(a) (i) $\frac{d}{dx}(2e^x - 4)^9 = 9e^x(2e^x - 4)^8$

(ii) $\frac{d}{dx}(x^2 \sin x) = x^2 . \cos x + \sin x . 2x$

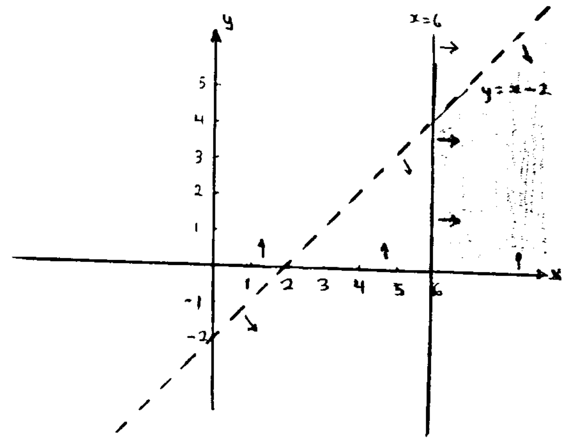
$= x^2\cos x + 2x\sin x$

(b) $\angle ACD = 55°$ (alt. ∠'s =, AB ∥ DC)

$\angle DPC = 180° - 55° - 36°$

$= 89°$ (∠ sum of Δ = 180°)

(c)

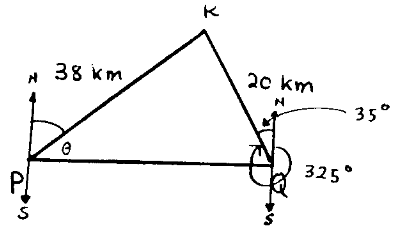


(d)(i) $\int \frac{2x}{x^2+5}\, dx = \ln(x^2+5) + c$

(ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\, dx = \left[\tan x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \sqrt{3} - 1$

(a)



(i) $\angle PQR = 90^\circ - 35^\circ = 55^\circ$

(ii) $\dfrac{a}{\sin a} = \dfrac{b}{\sin b}$ i.e. $\dfrac{20}{\sin\theta} = \dfrac{38}{\sin 55^\circ}$

$\sin\theta = \dfrac{20\sin 55^\circ}{38}$ i.e. $\theta = 25^\circ 32'$

Hence bearing of R from P is given by:

$90^\circ - (\theta) = 65^\circ$ to the nearest degree.

(b) (i) P(both spinners stop on same number)

$= P(\text{both } 1) + P(\text{both } 3)$

$= \frac{1}{3}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$

(ii) P(at least one stops on 3)

$= 1 - P(\text{neither stops on } 3)$

$= 1 - \frac{2}{3}\times\frac{3}{4} = 1 - \frac{1}{2} = \frac{1}{2}$

(c) (i) $x^2 - 4x = x - 4$

$x^5 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$\therefore x = 1, 4$

$\therefore$ point A has x-coordinate = 1

$y = 1 - 4 = -3$

$\therefore$ coordinates of A are $(1, -3)$

$$= \left|\int_1^4 x^2 - 4x \, dx\right| - \left|\int_1^4 x - 4 \, dx\right|$$

$$= \left|\left[\frac{x^3}{3} - 2x^2\right]_1^4\right| - \left|\left[\frac{x^2}{2} - 4x\right]_1^4\right|$$

$$= \left|\left(\frac{64}{3} - 32\right) - \left(\frac{1}{3} - 2\right)\right| - \left|(8-16) - \left(\frac{1}{2} - 4\right)\right|$$

$$= 9 - 4.5 = 4.5 \text{ units}^2$$

Question 5

(a) (i) $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

(ii) Let $f'(x) = 0$ to find s.p.

i.e. $4x^2(x-3) = 0$

$x = 0$ or $x = 3$

$f''(x) = 12x^2 - 24x$

$f''(3) = 108 - 72 = +36$

$\therefore x = 3, y = -27$ is a minimum t.p.

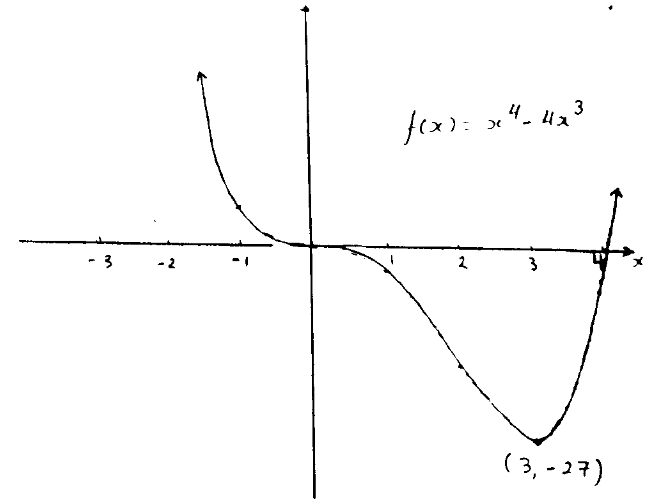
At $x = 0$, $f'(0) = 0$ and $f''(0) = 0$ $\therefore$ need to examine the sign of $f'(x)$ on either side of $x = 0$.

i.e.

x	-0.1	0	0.1
$f'(x)$	\	—	\

$\therefore x = 0, y = 0$ is a horizontal point of inflection.

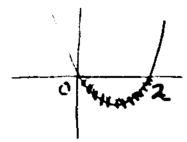
(iii)



(iv) $f(x)$ is concave down when $f''(x) < 0$

i.e. $12x^2 - 24x < 0$

$12x(x-2) < 0$



i.e. $0 < x < 2$

(b) (i) $T_n = a + d(n-1) = 180 - 3(n-1)$

$T_{20} = 180 - 3\times 19 = 123$ blocks

(ii) $T_n = a + d(n-1) = 123 - 1(n-1)$

Top row = 10 block

$\therefore$ number of rows above row 20 is:

$10 = 123 - n + 1$ i.e. $n = 114$ rows

$\therefore$ Total number of rows $= 20 + 114$

$= 134$ rows

(iii) $S_n = \frac{n}{2}(a + \ell)$

$= \frac{20}{2}(180 + 123)$ [rows 1 - 20]

$+ \frac{114}{2}(122 + 10)$ [rows 21 - 134]

$= 3030 + 7524 = 10{,}554$ blocks

(a) $\log_2(3x-4) = 5$

$3x - 4 = 2^5 = 32$

$3x = 36$

$x = 12$

(b) (i) Alt. $\angle$'s $=$, $BE \parallel CD$ $\therefore$ $\angle EBD = \angle BDC$

(ii) $\angle BCD = \angle ABE$ (corresp. $\angle$'s $=$, $BE \parallel CD$)

$\angle BDC = \angle BCD = \angle ABE$

$\therefore$ $\triangle BCD$ is isosceles (base $\angle$'s $=$)

(iii) $AB : BC = AE : ED$ $\left\{ \begin{array}{l} \text{parallel lines cutting} \\ \text{transversals will preserve} \\ \text{the ratio of the sides} \end{array} \right\}$

Also $BC = BD$ (sides of isos. $\triangle$ $=$)

Hence $AB : BD = AE : ED$

(c) (i) $C = C_0 e^{-kt}$

At $t = 0$, $C = 5$ i.e. $C_0 = 5$

Hence $C = 5e^{-kt}$

Let $C = 2.8$ to find k at $t = 1$

i.e. $2.8 = 5e^{-k}$

$e^{-k} = \frac{2.8}{5} = 0.56$

$k = -\ln(0.56) = 0.58$ to 2dp

(ii) Let $C = 0.2$, to find t

$0.2 = 5e^{-kt}$

$t = \frac{\ln(0.1)}{-k} = 3.97\ldots$

$\therefore$ Farmer must wait 4.0 years after the accident before the paddock can be used.

(a) (i) $S_\infty = \frac{a}{1-r} = \frac{2}{1 - \frac{1}{\sqrt{2}+1}}$

$= \frac{2(\sqrt{2}+1)}{\sqrt{2}} = 2 + \sqrt{2}$

(ii) A series has a limiting sum if $|r| < 1$

now $r = \frac{T_2}{T_1} = \frac{\frac{2}{\sqrt{2}-1}}{2} = \frac{1}{\sqrt{2}-1} = 2.4\ldots$

$\therefore$ Series has no limiting sum.

(b) (i) Particle at rest when $v = 0$

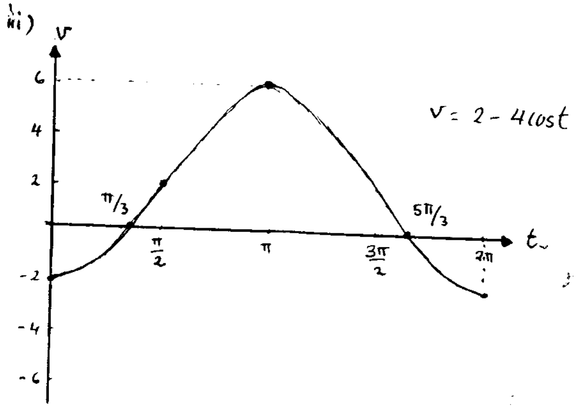
i.e. $2 - 4\cos t = 0$

$\cos t = \frac{1}{2}$

$t = \frac{\pi}{3}, \frac{5\pi}{3}$ seconds

(ii) $-1 \le \cos t \le 1$, $\therefore$ maximum velocity occurs when $\cos t = -1$ i.e. $v = 6$ m/s

(iii)



(iv) $\left| \int_0^{\pi/3} (2 - 4\cos t)\, dt \right| + \int_{\pi/3}^{\pi} (2 - 4\cos t)\, dt$

$= \left| \left[2t - 4\sin t \right]_0^{\pi/3} \right| + \left[2t - 4\sin t \right]_{\pi/3}^{\pi}$

$= \left| \frac{2\pi}{3} - 2\sqrt{3} \right| + (2\pi - 0) - \left(\frac{2\pi}{3} - 2\sqrt{3} \right)$

$= 2\sqrt{3} - 2\pi/3 + 2\pi - 2\pi/3 + 2\sqrt{3} = \frac{2\pi}{3} + 4\sqrt{3}$ m

(a) $y = 2$

(b) $V = \int_1^5 \pi x^2\, dy$ $\quad y = e^x \rightarrow x = \ln y$

$= \pi \int_1^5 (\ln y)^2\, dy$

(c)

x	2	4	6
$\frac{x}{\ln x}$	$\frac{2}{\ln 2}$	$\frac{4}{\ln 4}$	$\frac{6}{\ln 6}$

$\int_2^6 \frac{x}{\log_e x}\, dx \approx \frac{h}{3}(y_0 + y_2 + 4y_1)$

$= \frac{2}{3}\left(\frac{2}{\ln 2} + \frac{6}{\ln 6} + 4 \cdot \frac{4}{\ln 4} \right)$

$= 11.9$ to 1 d.p.

(d)(i) $x^2 = 12y$, $y = mx - 3m^2$

$x^2 = 12(mx - 3m^2) = 12mx - 36m^2$

i.e. $x^2 - 12mx + 36m^2 = 0$

$x = \frac{12m \pm \sqrt{144m^2 - 4 \cdot 1 \cdot 36m^2}}{2}$

$= \frac{12m}{2} = 6m$

$\therefore$ line $y = mx - 3m^2$ cuts parabola $x^2 = 12y$ for all values of m.

(ii) $y = mx - 3m^2$, $(5, 2)$

$2 = 5m - 3m^2 \rightarrow 3m^2 - 5m + 2 = 0$

$(3m - 2)(m - 1) = 0$

$m = \frac{2}{3}$, $m = 1$

(iii) From (i) $b^2 - 4ac = 0$

$\therefore$ $y = mx - 3m^2$ is tangent to $x^2 = 12y$

Equation of tangents passing through $(5,2)$ are:

- $m = 1$: $y = x - 3$
- $m = \frac{2}{3}$: $y = \frac{2}{3}x - 3\left(\frac{2}{3}\right)^2 = \frac{2}{3}x - \frac{12}{9}$

(a) $2\sin^2 x - 3\sin x - 2 = 0$

$(2\sin x + 1)(\sin x - 2) = 0$

$\sin x = -\frac{1}{2} \rightarrow x = \left(\pi + \frac{\pi}{6}\right), \left(2\pi - \frac{\pi}{6}\right)$

$= \frac{7\pi}{6}, \frac{11\pi}{6}$

$\sin x \neq 2 \therefore$ no solutions

$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}$

(b) (i) $a^2 = b^2 + c^2 - 2bc\cos A$

$r^2 = r^2 + 3r^2 - 2.r.r\sqrt{3}\cos A$

$2\sqrt{3}r^2\cos A = 3r^2$

$\cos A = \frac{3r^2}{2\sqrt{3}r^2} = \frac{\sqrt{3}}{2}$

$A = 30^\circ$

(ii) Area of sector BAC $= \frac{1}{2}r^2\theta_{rad}$.

$= \frac{1}{2}(r\sqrt{3})^2 \frac{\pi}{3}$

$= \frac{\pi r^2}{2}$ units^2

(iii) Area of sector BOC $= \frac{1}{2}r^2\theta_{rad}$.

$= \frac{1}{2}r^2 . \frac{2\pi}{3}$

$= \frac{\pi r^2}{3}$ units^2

(iv) Shaded Region

= Area of sector BOC − Area of sector BAC + 2 × △AOB

$= \frac{\pi r^2}{3} - \frac{\pi r^2}{2} + 2(r)(r\sqrt{3})\sin 30^\circ$

$= \sqrt{3}r^2 - \frac{\pi r^2}{6}$

$= \frac{r^2}{6}(6\sqrt{3} - \pi)$ units^2

$t = \frac{\text{distance}}{\text{speed}} = \frac{L}{(v-\mu)}$

i.e. $E = \frac{aLv^3}{(v-\mu)}$

(ii) $E = \frac{aLv^3}{v-\mu}$

$\frac{dE}{dv} = \frac{(v-\mu).3aLv^2 - aLv^3.1}{(v-\mu)^2}$

$= \frac{aLv^2[3v - 3\mu - v]}{(v-\mu)^2}$

$= \frac{aLv^2[2v-3\mu]}{(v-\mu)^2}$

Let $\frac{dE}{dv} = 0$ to find maxima/minima

i.e. $2v - 3\mu = 0 \rightarrow v = \frac{3}{2}\mu$

v	1.4μ	$3/2\mu$	1.6μ
$\frac{dE}{dv}$	−ve	0	+ve
	\	—	/

$\therefore$ minimum exists at $v = \frac{3}{2}\mu$

Question 10

(i) $A_2 = \left[120000\left(1+\frac{0.06}{k}\right) - F\right]\left(1+\frac{0.06}{k}\right) - F$

$= 120000\left(1+\frac{0.06}{k}\right)^2 - F\left[1 + \left(1+\frac{0.06}{k}\right)\right]$

(ii) Let $\alpha = 1 + \frac{0.06}{k} \rightarrow \alpha - 1 = \frac{0.06}{k}$

$A_n = 120000\left(1+\frac{0.06}{k}\right)^n - F\left[1 + \left(1+\frac{0.06}{k}\right) + \cdots + \left(1+\frac{0.06}{k}\right)^{n-1}\right]$

$= 120000\alpha^n - F\left[\frac{\alpha^n - 1}{\alpha - 1}\right]$

$\left[\frac{0.06}{k}\right]$

$= 120000\alpha^n - \frac{kF(\alpha^n - 1)}{0.06}$

(iii) $n = 25 \times 4 = 100$, $k = 4$, $\alpha = 1.015$

$120000(1.015)^{100} - \frac{4F(1.015^{100} - 1)}{0.06} = 0$

$F = \frac{120000(1.015)^{100} \times 0.06}{4(1.015^{100} - 1)} = \$2,324.47$ per quarter

(iv) $n = 25 \times 12 = 300$, $k = 12$, $\alpha = 1.005$

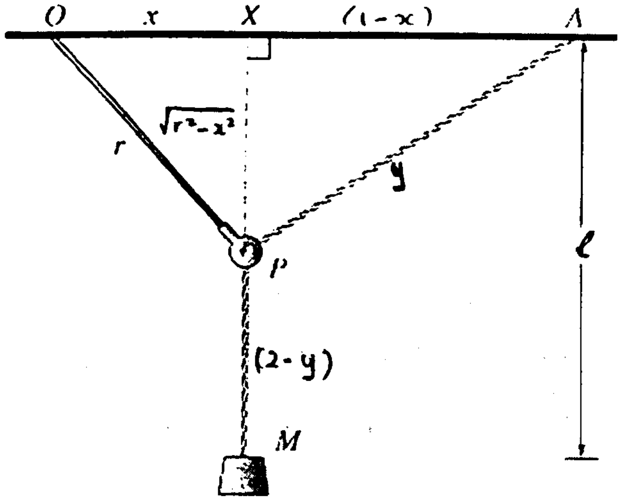
$120000(1.005)^{300} - \frac{12F(1.005^{300} - 1)}{0.06} = 0$

$F = \frac{120000(1.005)^{300} \times 0.06}{12(1.005^{300} - 1)} = \$773.16$ per month

Saving of the term of the loan

$= 2324.47 \times 100 - 773.16 \times 300$

$= \$499$



(b)

(i) Looking at ΔOXP

$XP^2 = r^2 - x^2$

i.e. $XP = \sqrt{r^2 - x^2}$

Let $AP = y$ and $PM = (2-y)$

Looking at ΔAXP

$(AX)^2 + (XP)^2 = y^2$

$(1-x)^2 + \left(\sqrt{r^2-x^2}\right)^2 = y^2$

$(1-2x+x^2) + r^2 - x^2 = y^2$

$1 - 2x + r^2 = y^2$

$y = \sqrt{1-2x+r^2}$ (since $y > 0$)

$\therefore \ l = XP + PM$

$= \sqrt{r^2-x^2} + 2 - \sqrt{1-2x+r^2}$

$= 2 + \sqrt{r^2-x^2} - \sqrt{1-2x+r^2}$

(ii) $l = 2 + \sqrt{r^2-x^2} - \sqrt{1-2x+r^2}$

$$\frac{dl}{dx} = \frac{1}{2}(r^2-x^2)^{-1/2} \cdot -2x - \frac{1}{2}(1-2x+r^2)^{-1/2} \cdot -2$$

$$= \frac{-x}{\sqrt{r^2-x^2}} + \frac{1}{\sqrt{1-2x+r^2}}$$

$$= \frac{-x\sqrt{1-2x+r^2} + \sqrt{r^2-x^2}}{\sqrt{r^2-x^2}\sqrt{1-2x+r^2}} \times \frac{\sqrt{r^2-x^2} + x\sqrt{1-2x+r^2}}{\sqrt{r^2-x^2} + x\sqrt{1-2x+r^2}}$$ (rationalising the denominator)

$$= \frac{(r^2-x^2) - x^2(1-2x+r^2)}{\sqrt{r^2-x^2}\sqrt{1-2x+r^2}\left[\sqrt{r^2-x^2} + x\sqrt{1-2x+r^2}\right]}$$

(iii) Let $\frac{dl}{dx} = 0$ and solve for x to find maxima / minima

i.e. $(x-1)(2x^2 - r^2x - r^2) = 0$

i.e. $x = 1$ or $2x^2 - r^2x - r^2 = 0$

$$x = \frac{r^2 \pm \sqrt{r^4 + 8r^2}}{4} = \frac{r^2 \pm r\sqrt{r^2+8}}{4}$$

since $x > 0$ $\therefore$ $x = 1$ or $x = \frac{r^2 + r\sqrt{r^2+8}}{4}$

However since $r < 1$, then $x \neq 1$

$\therefore$ $x_1 = \frac{r^2 + r\sqrt{r^2+8}}{4}$ is the only solution.

M is closest to the floor when l is maximised.

x	$x_1 - \varepsilon$	x_1	$x_1 + \varepsilon$
$\frac{dl}{dx}$	+ve	0	−ve

[Note: since $(x_1 - 1) < 0$ as $OA = 1\,m$ and $x \leq r < 1$]

$\therefore$ maximum occurs at $x_1 = \frac{r^2 + r\sqrt{r^2+8}}{4}$

Hence M is closest to the floor for $x = \frac{r^2 + r\sqrt{r^2+8}}{4}$