HSC 2005 MATHEMATICS (2 unit) EXAM : ANSWERS/SOLUTIONS.

Jan Hansen, janh@hansendata.com.au

Question 1

Q1.a Using a calculator $\sqrt{\dfrac{275.4}{5.2 \times 3.9}} = 3.7$ correct to two significant figures.
Q1.b $x^3 - 27 = x^3 - 3^3 = (x-3)(x^2 + 3x + 9)$
Q1.c $\displaystyle\int 4 + \sec^2 x \, dx = 4x + \tan x + c$
Q1.d $\dfrac{2x-3}{2} - \dfrac{x-1}{5} = \dfrac{5(2x-3) - 2(x-1)}{10} = \dfrac{8x-13}{10}$
Q1.e $|x-3| \leq 1$ can be rewritten as $-1 \leq x - 3 \leq 1$ which simplifies to give $2 \leq x \leq 4$.
Q1.f Comparing $x^2 = 8(y-1)$ to the standard form $(x-h)^2 = 4a(y-k)$ we have that $4a = 8$ so that the focal length is $a = 2$ and the vertex is $(h,k) = (0,1)$ so the focus which is a distance $a = 2$ vertically above the vertex has coordinates $(0,3)$.

Question 2

Q2.a The related angle is $\pi/4$ and recall using ASTC that cos is positive in quadrants 1 and 4. Hence $\theta = \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4} = \dfrac{\pi}{4}, \dfrac{7\pi}{4}$
Q2.b
Q2.b.i Using the product rule, $\dfrac{d}{dx}(x \sin x) = x \cos x + 1. \sin x = x \cos x + \sin x$.
Q2.b.ii Using the quotient rule, $\dfrac{d}{dx}\left(\dfrac{x^2}{x-1}\right) = \dfrac{(x-1).2x - x^2.1}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$.
Q2.c
Q2.c.i $I = \displaystyle\int \frac{6x^2}{x^3+1} \, dx$. Let $u = x^3 + 1$, $\dfrac{du}{dx} = 3x^2$ so the derivative of the denominator is equal to half the numerator. We thus have a log situation.
Recall that for a function $f(x)$, $\dfrac{d}{dx} \ln f(x) = \dfrac{f'(x)}{f(x)}$.
Hence $\displaystyle\int \frac{f'(x)}{f(x)} \, dx = \ln f(x) + c$.
In this case, after adjusting the constant, we have $I = 2\ln(x^3 + 1) + c$
Q2.c.ii $I = \displaystyle\int_0^{\pi/6} \cos 3x \, dx = \left[\frac{\sin 3x}{3}\right]_0^{\pi/6} = \frac{\sin(3.\pi/6)}{3} - 0 = \frac{\sin(\pi/2)}{3} = \frac{1}{3}$.
Q2.d We have $y = \ln x$. Differentiating,
$y' = \dfrac{1}{x}$ and so evaluating at the point $(e, 1)$ we have, the gradient of the tangent is $m = \dfrac{1}{e}$ and the equation is $y - 1 = \dfrac{1}{e}(x - e)$ which simplifies to $x - ey = 0$.

Question 3

Q3.a $\displaystyle\sum_{n=3}^{5}(2n+1) = (2.3+1)+(2.4+1)+(2.5+1) = 7+9+11 = 27.$

Q3.b

Q3.b.i Using the cosine rule, $\cos\theta = \dfrac{7^2+8^2-13^2}{2.7.8} = -\dfrac{1}{2}$ and so $\theta = \dfrac{2\pi}{3}$.

Q3.b.ii Area $= \dfrac{1}{2}.7.8.\sin\dfrac{2\pi}{3} = 28.\sqrt{3}/2 = 14\sqrt{3}$ units^2.

Q3.c

Q3.c.i AD has the same gradient as BC, so find the gradient of BC, $m_{BC} = \dfrac{6-0}{12-9} = 2$. The line AD passes through $A(6,0)$ so has equation $y-0 = 2(x-6)$ or $y = 2x-12$.

Q3.c.ii D is the intersection of the lines $y = 2x-12$ and $x-2y = 0$ and so we have $y = 2.2y-12$. Solving we find, $y = 4$ and so $D(8,4)$.

Q3.c.iii By inspection we notice that E has the same y value as D and its x value is 2 more than B (since we have a parallelogram) and hence $E(9+2,4) = E(11,4)$.

Q3.c.iv $\angle DOA = \angle CDE$ (since $DE \parallel AB$) and
$\angle ODA = \angle DCE$ (since $AD \parallel BE$)
and $\therefore$ $\angle OAD = \pi - \angle DOA - \angle ODA = \pi - \angle CDE - \angle DCE = \angle DEC$ that is $\angle OAD = \angle DEC$ and we have AAA (three angles euqal) so that $\Delta OAD \,|||\, \Delta DEC$ as required.

Q3.c.v By the similarity of triangles from part (iv) we have
$\dfrac{AD}{EC} = \dfrac{OA}{DE} = \dfrac{6}{3} = 2$ or $AD : EC = 2:1$

Question 4

Q4.a

Q4.a.i The arc AB has length $l = r\theta = 90\times 0.6 = 54$ cm.

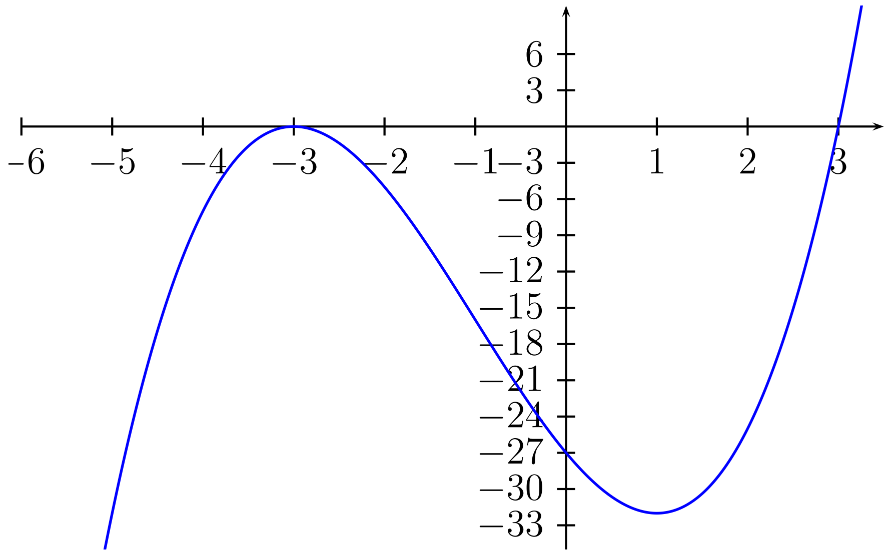
Q4.a.ii Using the cosine rule and denoting the straight line distance between A and B as d we have, $d^2 = 90^2+90^2-2.90.90.\cos 0.6 = 2829.563039$. So $d = 53.2$ cm (to one decimal place).

Q4.a.iii Area $= \dfrac{1}{2}r^2\theta = \dfrac{1}{2}90^2\times 0.6 = 2430$ cm^2.

Q4.b

Q4.b.i $f(x) = (x+3)(x^2-9) = (x+3)^2(x-3) = 0$ has solutions $x = -3,-3,3$. (-3 is a double root)

Q4.b.ii $f'(x) = (x+3).2x+1.(x^2-9) = 3x^2+6x-9 = 3(x^2+2x-3) = 3(x-1)(x+3)$. Also, $f''(x) = 6x+6 = 6(x+1)$.
For stationary points we put $f'(x) = 0$ to obtain two stationary points at $x = 1,-3$. So, $(1,-32)$, $(-3,0)$. Since $f''(1) > 0, f''(-3) < 0$ we know that $(1,-32)$ is a minimum turning point and $(-3,0)$ is a maximum turning point.



Q4.b.iii

Q4.b.iv Concave down is when the second derivative is negative so solve $f''(x) = 6(x + 1) < 0$ which gives $x < -1$.

Question 5

Q5.a $\log_3 7 = \dfrac{\ln 7}{\ln 3} = 1.77$ correct to two decimal places.

Q5.b

$BC \parallel AD$, $AB \parallel DC$, and $BC = AD$ (Given $ABCD$ is a parallelogram)
$\angle ADC = 60^o$ (cointerior since $AB \parallel DC$)
$\angle BCE = \angle ADC$ (since $AD \parallel BC$)
$\therefore \angle BCE = 60^o$
Since ΔBCE is isosceles then $\angle BEC = 60^o$ also.
Hence $\angle CBE = 60^o$ (sum angles in a Δ)
Hence ΔBCE is equilateral as required.

Q5.c The curve $y = 2e^x + 3x$ has gradient $y' = 2e^x + 3$ and so tangents which are parallel to the line $y = 5x - 3$ satisfy $2e^x + 3 = 5$ and hence $e^x = 1$ giving $x = 0$ and $P(0, 2)$.

Q5.d

Q5.d.i $P(RRR) = \dfrac{100}{300} \times \dfrac{99}{299} \times \dfrac{98}{298} = \dfrac{1617}{44551}$.

Q5.d.ii $P(\text{at least one not red}) = 1 - P(RRR) = 1 - \dfrac{1617}{44551} = \dfrac{42934}{44551}$

Q5.d.iii

The probability that, for instance, there is one winning ticket of each in the particular order RGB equals $\dfrac{100}{300} \times \dfrac{100}{299} \times \dfrac{100}{298} = \dfrac{5000}{44551}$.
But the possible ways of having one of each is 3!, the possibilities being;
$BGR, BRG, GBR, GRB, RBG, RGB$
Hence the probability that there is one winning ticket of each color is
$3! \times \dfrac{100}{300} \times \dfrac{100}{299} \times \dfrac{100}{298} = \dfrac{20000}{44551} = \dfrac{2500}{11063}$.

Question 6

Q6.a $\int_0^{20} f(x)\,dx = \frac{h}{3}[y_0+y_4+4(y_1+y_3)+2y_2] = \frac{5}{3}[15+10+4(25+18)+2(22)] = 401.7$ or 402 rounded up.

Q6.b Firstly, the given conditions are when $t = 60$ min, $V = 0$ L, and when $t = 0$ min, $V = 3600$ L.

Q6.b.i $V(10) = 3600\left(1 - \frac{10}{60}\right)^2 = 2500$ L.

Q6.b.ii $\frac{dV}{dt} = 7200(1 - t/60) \times -1/60 = -120\left(1 - \frac{t}{60}\right)$

So, $\frac{dV}{dt}(20) = -120(1 - 20/60) = -80$.

After 20 minutes the water drains from the tank at the rate of 80 L/min.

Q6.b.iii $\frac{dV}{dt} = 2t - 120$

We want to know when $120 - 2t$ is a maximum for $0 \le t \le 60$, and this occurs when $t = 0$.

Q6.c

Q6.c.i Solve $x^2 = 12 - 2x^2$. Gives $3x^2 = 12$, $x^2 = 4$ $x = \pm 2$. The two points of intersection are thus $(-2, 4), (2, 4)$.

Q6.c.ii

$$\begin{aligned}
V &= \pi \int_0^4 x_1^2\,dy + \pi \int_4^{12} x_2^2\,dy \\
&= \pi \int_0^4 y\,dy + \pi \int_4^{12} \frac{12-y}{2}\,dy \\
&= \frac{\pi}{2}\left[y^2\right]_0^4 + \pi\left[6y - y^2/4\right]_4^{12} \\
&= \frac{\pi}{2}.16 + \pi\left(6.12 - 12^2/4\right) - \pi\left(6.4 - 4^2/4\right) \\
&= 24\pi
\end{aligned}$$

Question 7

Q7.a

For Anne,

$$\begin{aligned}
A_1 &= 50000 \\
A_2 &= 50000 + 2500 \\
&\vdots \\
A_n &= 50000 + (n-1)2500
\end{aligned}$$

For Kay,

$$\begin{aligned}
K_1 &= 50000 \\
K_2 &= 50000 \times 1.04 \\
&\vdots \\
K_n &= 50000 \times 1.04^{n-1}
\end{aligned}$$

Q7.a.i $A_{13} = 50000 + (13-1)2500 = \$80,000$

Q7.a.ii $K_{13} = 50000 \times 1.04^{12} = \$80,051.61$

Q7.a.iii

Method 1

Let T_n, S_n be the total amounts paid to Kay, Anne in their first N years respectively. Then

$$\begin{aligned}
T_1 &= K_1 = 50000\\
T_2 &= K_1 + K_2 = 50000 + 50000 \times 1.04\\
T_3 &= 50000 + 50000 \times 1.04 + 50000 \times 1.04^2\\
&\vdots\\
T_n &= 50000(1 + 1.04 + 1.04^2 + \cdots + 1.04^{n-1})\\
T_n &= 50000(\frac{1.04^n - 1}{0.04})\\
T_n &= 50000(\frac{1.04^{20} - 1}{0.04}) = \$1,488,903.93
\end{aligned}$$

and

$$\begin{aligned}
S_1 &= A_1 = 50000\\
S_2 &= A_1 + A_2 = 50000 + 50000 + 2500\\
S_3 &= 50000 + (50000 + 2500) + (50000 + 2 \times 2500)\\
&\vdots\\
S_n &= 50000 \times n + [2500 + 2 \times 2500 + \cdots + (n-1) \times 2500)\\
S_n &= 50000 \times n + \left(\frac{n-1}{2}\right)(2500 + (n-1)2500)\\
S_n &= 50000 \times n + 1250n - 1n\\
S_{20} &= 50000 \times 20 + 12501920 = \$1,475,000
\end{aligned}$$

Thus $T_{20} - S_{20} = \$1,488,903.93 - \$1,475,000 = \$\mathbf{13,903.93}$

Method 2 (More for 3 unit students comfortable with summation notation)

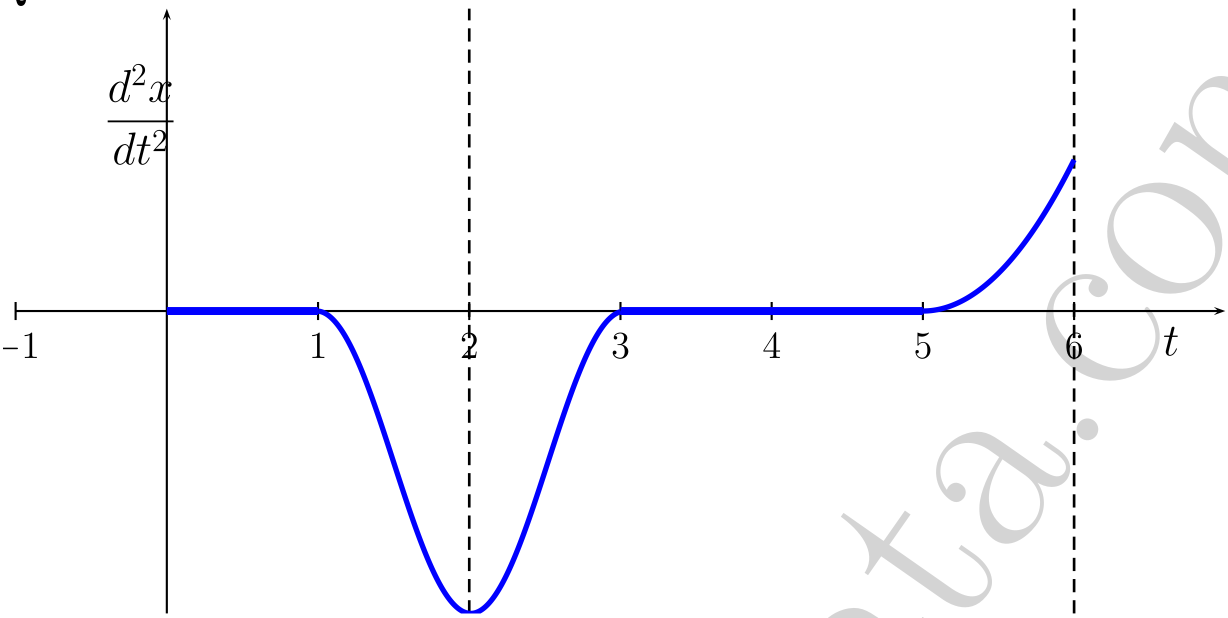
$$\begin{aligned}
T_n - S_n &= \sum_{n=1}^{N} K_n - \sum_{n=1}^{N} A_n\\
T_n - S_n &= \sum_{n=1}^{N}(50000 \times 1.04^{n-1}) - \sum_{n=1}^{N}(50000 + (n-1)2500)\\
T_n - S_n &= 50000\sum_{n=1}^{N} 1.04^{n-1} - \sum_{n=1}^{N}(47500 + 2500n)\\
T_n - S_n &= 50000\sum_{n=1}^{N} 1.04^{n-1} - 47500\sum_{n=1}^{N} 1 - 2500\sum_{n=1}^{N} n\\
T_n - S_n &= 50000(\frac{1.04^N - 1}{0.04}) - 47500N - 1250N((1+N)\\
T_n - S_n &= 50000(\frac{1.04^N - 1}{0.04}) - 48750N - 1250N^2\\
T_{20} - S_{20} &= 50000(\frac{1.04^{20} - 1}{0.04}) - 48750 \times 20 - 1250 \times (20)^2\\
T_{20} - S_{20} &= 1,488,903.93 - 975,000 - 500,000\\
T_{20} - S_{20} &= \$\mathbf{13,903.93}
\end{aligned}$$

Q7.b Initial conditions: $t = 0$, $x = 0$.

Q7.b.i When $t = 2$.

Q7.b.ii When $t = 4$. The symmetry of the curve about $t = 2$ shows the way out is exactly reversed on way in up to $t = 4$ so that is when the particle returns.

Q7.b.iii



Question 8

Q8.a

Q8.a.i

Pythagoras says $h^2 + x^2 = R^2$ so $x^2 = R^2 - r^2$.
Volume, $V = \pi x^2.2h = 2\pi h(R^2 - r^2)$ as required.

Q8.a.ii

$\dfrac{dV}{dx} = 2\pi R^2 - 6\pi h^2 = 0$ for stationary points.

$\therefore R^2 = 3h^2$ and $h = \dfrac{R}{\sqrt{3}}$.

$\dfrac{d^2V}{dx^2} = -12\pi h$ and so $\dfrac{d^2V}{dx^2}(\dfrac{R}{\sqrt{3}}) < 0$ so its concave down and and shows that a maximum volume occurs when $h = \dfrac{R}{\sqrt{3}}$.

Q8.b Area $= \dfrac{\pi.2^2}{4} - \displaystyle\int_0^1 (x^2 - 3x + 2)\,dx = \pi - \left(\dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x\right)_0^1 = \pi - (\dfrac{1}{3} - \dfrac{3}{2} + 2) = \pi - \dfrac{5}{6}$.

Q8.c

Q8.c.i

$$
\begin{aligned}
A_1 &= 3 \times 10^6 \times 1.12 - 4.8 \times 10^5 \\
A_2 &= A_1 \times 1.12 - 4.8 \times 10^5 \\
A_2 &= 3 \times 10^6 \times (1.12)^2 - 4.8 \times 10^5 \times 1.12 - 4.8 \times 10^5 \\
A_2 &= 3 \times 10^6 \times (1.12)^2 - 4.8 \times 10^5(1 + 1.12)
\end{aligned}
$$

Q8.c.ii

Following on from the previous part,

$$\begin{aligned}
A_1 &= 3\times 10^6\times 1.12 - 4.8\times 10^5\\
A_2 &= 3\times 10^6\times (1.12)^2 - 4.8\times 10^5(1+1.12)\\
&\vdots\\
A_n &= 3\times 10^6\times (1.12)^n - 4.8\times 10^5(1+1.12+(1.12)^2+\cdots+(1.12)^{n-1})\\
&= 3\times 10^6\times (1.12)^n - 4.8\times 10^5\left(\frac{(1.12)^n-1}{1.12-1}\right)\\
&= 3\times 10^6\times (1.12)^n - \frac{4.8\times 10^5}{0.12}\times(1.12)^n + \frac{4.8\times 10^5}{0.12}\\
&= 10^6\times(1.12)^n(3-4.8/1.2)+40\times 10^5\\
&= 10^6\times(1.12)^n(-1)+4\times 10^6\\
&= 10^6\left[4-(1.12)^n\right]
\end{aligned}$$

as required.

Q8.c.iii

Upon final repayment the amount owing will be zero, so we solve $A_n = 10^6\left[4-(1.12)^n\right] = 0$ for n. That is,

$$\begin{aligned}
1.12^n &= 4\\
\ln 1.12^n &= \ln 4\\
n\ln 1.12 &= \ln 4\\
n &= \frac{\ln 4}{\ln 1.12} = 12.2325
\end{aligned}$$

Thus $n = 13$ and since $n = 1$ corresponds to the year 2005 then the year the final repayment is made is **2017**.

Question 9

Q9.a

The given conditions: $t = 0, x = 0, \dot{x} = 0$ will be used to find the constant.

Q9.a.i

$\ddot{x} = 4\sin 2t$

$\dot{x} = \dfrac{4\cos 2t}{-2} + c$

$\dot{x} = c - 2\cos 2t$

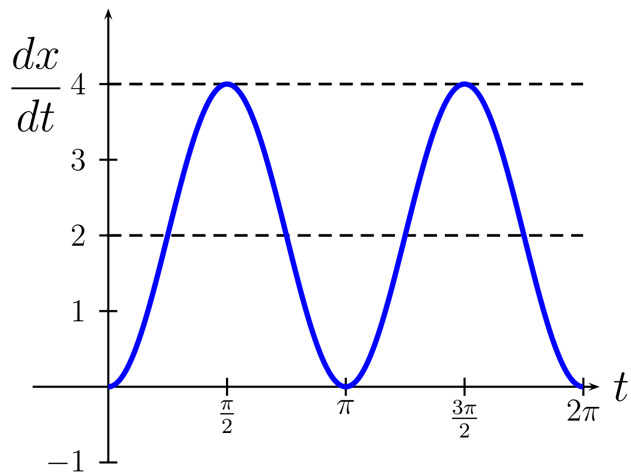
$0 = c - 2\cos 0$

$c = 2$

So, $\dot{x} = 2 - 2\cos 2t$,

as required.

Q9.a.ii



The particle comes to rest again when $t = \pi$.

Q9.a.iii

Distance travelled $= \int_0^\pi (2 - 2\cos 2t)\,dt = [2t - \sin 2t]_0^\pi = \mathbf{2}\pi.$

Q9.b

Q9.b.i Firstly,
$\angle FDE = \theta$ and $\angle ABD = \angle EDB$ (since $DE \parallel AB$).
$\therefore \angle DBE = 90^o - \angle EDB = 90^o - \angle ABD = \theta.$
In ΔADB, $\sin\theta = BD/6$, so $BD = 6\sin\theta$.
In ΔEDB, $\sin\theta = \dfrac{DE}{BD} = \dfrac{DE}{6\sin\theta}$
so $DE = 6\sin^2\theta$
In ΔEDF, $\sin\theta = FE/DE = \dfrac{FE}{6\sin^2\theta}$ so $FE = 6\sin^3\theta$ as required.

Q9.b.ii

$BD + EF + GH + \cdots = 6\sin\theta + 6\sin^3\theta + 6\sin^5\theta + \cdots$
Limiting sum, $S_\infty = \dfrac{a}{1-r} = \dfrac{6\sin\theta}{1-\sin^2\theta} = \dfrac{6\sin\theta}{\cos^2\theta} = 6\tan\theta\sec\theta$ as required.

Question 10

Q10.a

Q10.a.i The parabola $y = x^2$ and the line $y = mx + b$ intersect when
$x^2 = mx + b$
$x^2 - mx - b = 0$
The roots are given as α, β and the sum, product of the roots gives,
$\alpha + \beta = \dfrac{-(-m)}{1} = m$
$\alpha\beta = \dfrac{-b}{1} = -b.$

Q10.a.ii Using the formula for distance between the points A, B we have (after squaring)

$$\begin{aligned}
AB^2 &= (\alpha^2 - \beta^2)^2 + (\alpha - \beta)^2 \\
&= [(\alpha - \beta)(\alpha + \beta)]^2 + (\alpha - \beta)^2 \\
&= (\alpha - \beta)^2 \left[1 + (\alpha + \beta)^2\right] \\
&= (\alpha - \beta)^2 \left[1 + m^2\right] \text{ by (i)} \\
(\alpha - \beta)^2 &= \alpha^2 + \beta^2 - 2(\alpha\beta) \\
&= (\alpha + \beta)^2 - 4(\alpha\beta) \\
&= m^2 - 4(-b) \\
&= m^2 + 4b \\
\therefore AB^2 &= (m^2 + 4b)\left[1 + m^2\right] \\
\therefore AB &= \sqrt{(m^2 + 4b)\,(1 + m^2)}
\end{aligned}$$

as required.

Q10.a.iii The area required is the difference between the trapezium with top points A, B minus the smaller trapezia with top points A, P and PB respectively,

$\text{Area} = \frac{1}{2}\left[(\beta-\alpha)(\alpha^2+\beta^2)-(\beta-x)(\beta^2+x^2)-(x-\alpha)(x^2+\alpha^2)\right]$
$= \frac{1}{2}\left[(\beta-\alpha)(\alpha^2+\beta^2)-(\beta^3-x\beta^2+\beta x^2-x^3)-(x^3-\alpha x^2+x\alpha^2-\alpha^3)\right]$
$= \frac{1}{2}\left[(\beta-\alpha)(\alpha^2+\beta^2)+\alpha^3-\beta^3+x(\beta^2-\alpha^2)+x^2(\alpha-\beta)\right]$
$= \frac{1}{2}\left[\beta\alpha^2-\alpha^3+\beta^3-\alpha\beta^2+\alpha^3-\beta^3+x(\beta-\alpha)(\beta+\alpha)-x^2(\beta-\alpha)\right]$
$= \frac{(\beta-\alpha)}{2}\left[-\alpha\beta+x(\beta+\alpha)-x^2\right]$
$= \frac{\sqrt{m^2+4b}}{2}\left(b+x\,m-x^2\right)$ [using the product and sum of roots results and the working from part (ii)]
as required.

Q10.a.iv

To maximise we will differentiate the area and put it equal to zero to find stationary points;
$A = \frac{1}{2}\left(b+x\,m-x^2\right)\sqrt{m^2+4b}$
$\frac{dA}{dx} = \frac{1}{2}\sqrt{m^2+4b}\,(m-2x) = 0$
Solving we have $x = m/2$ and
$\frac{d^2A}{dx^2} = -\sqrt{m^2+4b} < 0$ so its concave down and therefore a maximum occurs for this x value.

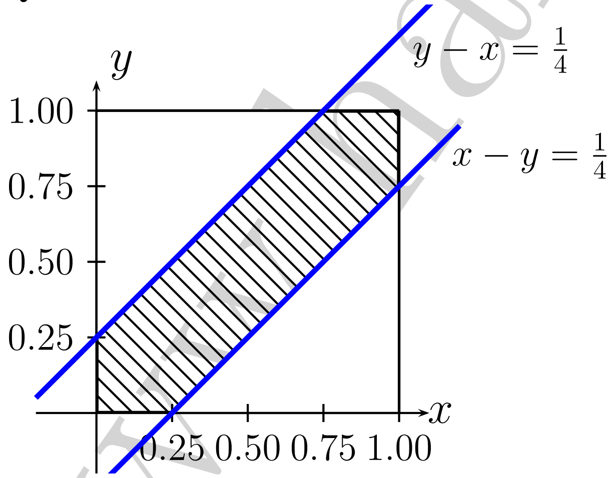
When $x = m/2$, $y = (m/2)^2 = m^2/4$ so the coordinates are $P(\frac{m}{2}, \frac{m^2}{4})$.

Q10.b

Q10.b.i

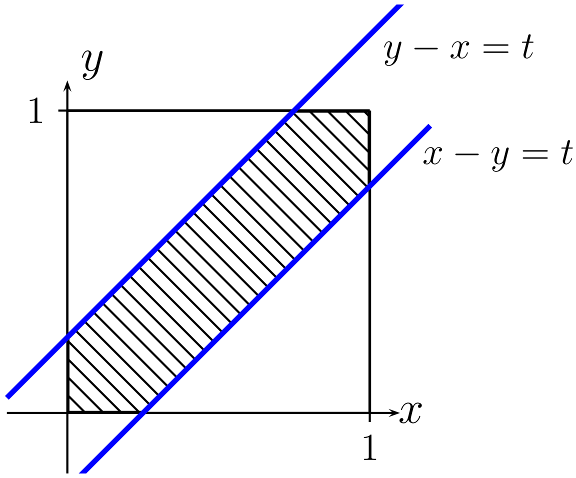
Since 1/4 represents $1/4 \times 60 = 15$ minutes.
So if x, y are within 15 minutes of each other then it is either $x - y \le \frac{1}{4}$ or $y - x \le \frac{1}{4}$.

Q10.b.ii



From the diagram we can see that the shaded area equals the square minus the two triangles, and the two triangles are a square of $\frac{3}{4} \times \frac{3}{4}$.
Therefore, the probability is
$p = 1 - \dfrac{3}{4} \times \dfrac{3}{4} = 1 - \dfrac{9}{16} = \dfrac{7}{16}$.

Q10.b.iii



$p(t) = 1 - (1 - t)^2 = 0.5$
$1 - (1 - 2t + t^2) = 0.5$
$t^2 - 2t + \dfrac{1}{2}$
$t = \dfrac{2 \pm \sqrt{4 - 4.1.1/2}}{2}$
$t = 1 \pm \dfrac{\sqrt{2}}{2}$
and since $0 \leq t \leq 1$ then we must have
$t = \dfrac{2 - \sqrt{2}}{2} = \dfrac{2 - \sqrt{2}}{2} \times 60\,\text{mins} = 0.292893 \times 60\,\text{mins} = 17$ minutes, 34 seconds or round up to 18 minutes.

The End