HSC 2008 MATHEMATICS (2 unit) EXAM : ANSWERS/SOLUTIONS.

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Question 1

Q1.a

Using a calculator in radian mode, $2\cos\frac{\pi}{5} = 1.62$ (2 d.p.)

(or in degree mode as $\cos\frac{180}{5} = \cos 36^o$)

Q1.b

$3x^2 + x - 2 = (3x-2)(x+1)$

Q1.c

$\dfrac{2}{n} - \dfrac{1}{n+1} = \dfrac{2\times(n+1) - n\times 1}{n(n+1)} = \dfrac{n+2}{n(n+1)}$

Q1.d

To solve $|4x-3| = 7$ we solve both $4x-3 = 7$ and $4x-3 = -7$. This gives $4x = 10$ or $4x = -4$ and finally, $x = -1, 2.5$.

Q1.e

$(\sqrt{3}-1)(2\sqrt{3}+5) = 2\times 3 - 2\sqrt{3} + 5\sqrt{3} - 5 = 1 + 3\sqrt{3}$

Q1.f

$S_n = \dfrac{n}{2}(2a + (n-1)d) = \dfrac{21}{2}(2\times 3 + (21-1)\times 4) = \dfrac{21}{2}(6+80) = 903.$

Question 2

Q2.a

Q2.a.i

$\dfrac{d}{dx}\left(x^2+3\right)^9 = 9\left(x^2+3\right)^8 \times 2x = 18x\left(x^2+3\right)^8$

Q2.a.ii

$\dfrac{d}{dx}x^2\log_e x = x^2.\dfrac{1}{x} + 2x.\log_e x = x + 2x\log_e x.$

Q2.a.iii

$\dfrac{d}{dx}\dfrac{\sin x}{x+4} = \dfrac{(x+4)\cos x - \sin x \times 1}{(x+4)^2} = \dfrac{(x+4)\cos x - \sin x}{(x+4)^2}$

Q2.b

The midpoint is $M(\dfrac{-1+5}{2}, \dfrac{4+8}{2}) = M(2,6)$.

The line is therefore,

$y - 6 = -\dfrac{1}{2}(x-2)$

$\therefore -2y + 12 = x - 2$

$\therefore x + 2y - 14 = 0$

Q2.c

Q2.c.i

$\displaystyle\int \frac{dx}{x+5} = \ln|x+5| + c.$

Q2.c.ii

$\displaystyle\int_0^{\pi/12} \sec^2 3x dx = \left[\frac{\tan 3x}{3}\right]_0^{\pi/12} = \frac{\tan\frac{\pi}{4}}{3} - 0 = \frac{1}{3}.$

Question 3

Q3.a

Q3.a.i The gradient of BC is $m_{BC} = \dfrac{5-3}{1-0} = 2$. The gradient of AD is 2 since the equation of the line is $y = 2x - 1$ and the coefficient of x is the gradient.

So $BC \parallel AD$ and hence $ABCD$ is a trapezium as two sides are parallel.

Q3.a.ii

D lies on the line $2x - y - 1 = 0$ so when $y = 5$, $2x - 5 - 1 = 0 \implies 2x = 6 \implies x = 3$, and hence $D(3,5)$.

Q3.a.iii

$|BC| = \sqrt{(5-3)^2 + (1-0)^2} = \sqrt{4+1} =$

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$\sqrt{5}$ units.

Q3.a.iv

Using the perpendicular distance formula we have,
$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right| = \left|\frac{2(0)-(3)-1}{\sqrt{2^2+1^2}}\right| = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$

Q3.a.v

Area $= \frac{1}{2} \times h \times (a+b) = \frac{1}{2} \times \frac{4}{\sqrt{5}} \times (\sqrt{5} + \sqrt{45}) = 2(1+3) = 8$ units^2.

Q3.b

Q3.b.i

$\frac{d}{dx}\log_e(\cos x) = \frac{1}{\cos x} \times -\sin x = -\tan x$.

Q3.b.ii

$\int_0^{\pi/4} \tan x\,dx = [-\ln(\cos x)]_0^{\pi/4} = -\ln(\cos \pi/4) + \ln(\cos 0) = -\ln(1/\sqrt{2}) + \ln 1 = \ln\sqrt{2}$.

Question 4

Q4.a

$\angle YXR = \angle XRQ$ (Alternate angles are equal in parallel lines $XY \parallel QR$.)
$\angle YRX = \angle XRQ$ (given).
$\therefore \Delta XYR$ is isosceles (two equal base angles).

Q4.b

Q4.b.i

$H = 50 \times 1.2^8 = 215$ mm.

Q4.b.ii

$50 \times 1.2^n > 400$
$1.2^n > 8$
$n \ln 1.2 > \ln 8$
$n > \ln 8 / \ln 1.2$
$n > 11.4$
$\therefore n = 12$

Q4.c

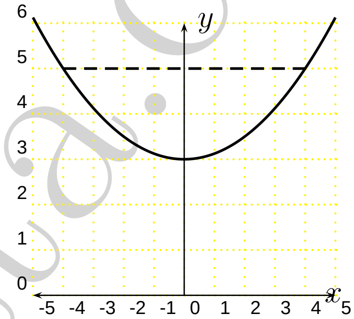
Q4.c.i

Vertex is $(0, 3)$.

Q4.c.ii

Solving $4a = 8$ gives $a = 2$. So the focus is $(0, 5)$.

Q4.c.iii



$x^2 = 8(y-3)$ where the focus is $F(0,5)$ and the vertex $(0,3)$

Q4.c.iv

Area $= \int_{-4}^{4} 5 - (x^2/8 + 3)dx = \left[2x - x^3/24\right]_{-4}^{4} = (8 - 64/24) - (-8 + 64/24) = 10\frac{2}{3}$

Question 5

Q5.a

Given $y' = 1 - 6\sin 3x$. Then $y = x + 2\cos 3x + c$ and using the given point $(0, 7)$ we have $7 = 0 + 2\cos 0 + c$, so that $c = 5$. Updating our function we have $\boxed{y = x + 2\cos 3x + 5}$

Q5.b

Q5.b.i

Firstly, $r = \frac{10x}{5} = 2x$. For a limiting sum to exist we must have $|r| < 1$. Hence we must solve $|2x| < 1$. This gives $|x| < 0.5$ and finally $-0.5 < x < 0.5$.

Q5.b.ii

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$S_\infty = \dfrac{a}{1-r} = \dfrac{5}{1-2x}$.
Now we must solve

$$\begin{aligned} \frac{5}{1-2x} &= 100 \\ 1-2x &= \frac{5}{100} \\ 2x &= \frac{95}{100} \\ x &= \frac{95}{200} = \frac{19}{40} \end{aligned}$$

Q5.c

Q5.c.i

When $s = 0$, $I = 6000$. $\therefore 6000 = Ae^0$ and so $A = 6000$ lux.

Q5.c.ii

$$\begin{aligned} 1000 &= 6000e^{-6k} \\ e^{-6k} &= \frac{1}{6} \\ -6k &= \ln(1/6) = \ln 1 - \ln 6 = -\ln 6 \\ \therefore k &= \frac{\ln 6}{6} \end{aligned}$$

Q5.c.iii

$$\begin{aligned} \frac{dT}{ds} &= A(-k)e^{-ks} \\ &= -Ake^{-ks} \\ &= -6000 \times \tfrac{\ln 6}{6} \times e^{-\frac{\ln 6}{6}\times 6} \\ &= -1000 \times \ln 6 \times \tfrac{1}{6} \\ &= \tfrac{-1000 \ln 6}{6} \\ &= 299 \text{ correct to nearest whole} \end{aligned}$$

Question 6

Q6.a

Firstly $-\pi \le x \le \pi \implies -\pi/3 \le x/3 \le \pi/3$.
$2\sin^2 \frac{x}{3} = 1$
$\sin^2 \frac{x}{3} = 0.5$
$\sin \frac{x}{3} = \pm 1/\sqrt{2}$.
Related angle is $\pi/4$. Hence $x/3 = -\pi/4, \pi/4$.
$\therefore x = -3\pi/4, 3\pi/4$.

Q6.b

Q6.b.i

When $t = 0$, $v = 20$ ms^{-1}.

Q6.b.ii

Velocity equals zero when $t = 10$ seconds.

Q6.b.iii

Acceleration is zero when $t = 6$ seconds.

Q6.b.iv

Distance travelled $= \dfrac{4-0}{6}(f(0) + 4f(2) + f(4)$
$\dfrac{8-4}{6}(f(4) + 4f(6) + f(8)) = \dfrac{2}{3}[20 + 4(50) + 7$
$= \boxed{493.3}$ m.

Q6.c

$$\begin{aligned} V &= \pi \int_3^6 y^2\,dx \\ &= \pi \int_3^6 \left(\frac{5}{x-2}\right)^2 dx \\ &= 25\pi \int_3^6 (x-2)^{-2}\,dx \\ &= 25\pi \left[\frac{(x-2)^{-1}}{-1}\right]_3^6 \\ &= 25\pi \left(\frac{4^{-1}}{-1} - \frac{1^{-1}}{-1}\right) \\ &= 25\pi \left(1 - \frac{1}{4}\right) \\ &= \frac{75\pi}{4} \end{aligned}$$

Question 7

Q7.a

$$\begin{aligned} \ln x - \frac{3}{\ln x} &= 2 \\ (\ln x)^2 - 3 &= 2\ln x \\ (\ln x)^2 - 2\ln x - 3 &= 0 \\ (\ln x + 1)(\ln x - 3) &= 0 \\ \therefore \ln x = -1 \quad &\text{or} \quad \ln x = 3 \end{aligned}$$

$x = e^3, \dfrac{1}{e}$

Q7.b

Q7.b.i

We have $\theta \le 2\pi$ and $\dfrac{\theta}{2\pi} = \dfrac{10\pi/3}{2\pi r} \implies \theta =$

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$\dfrac{10\pi}{3r}$ and so,

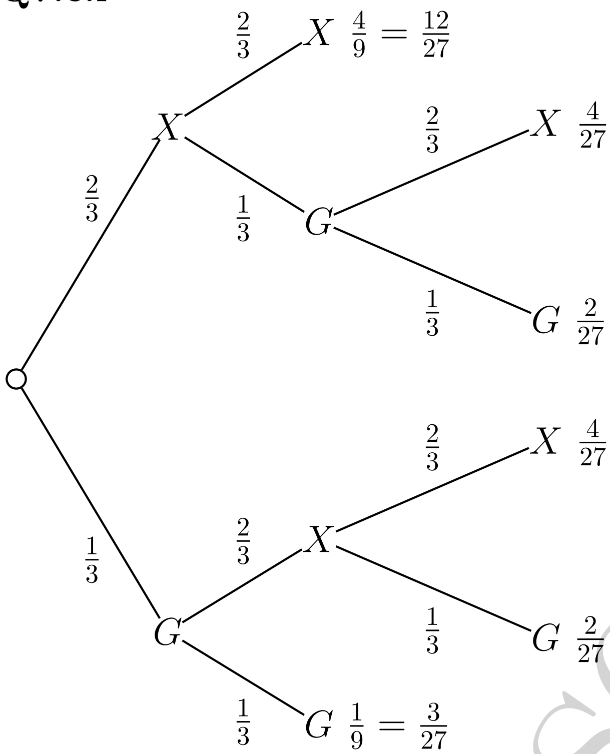
$\dfrac{10\pi}{3r} \le 2\pi \implies r \ge 5/3$

Q7.b.ii

$$\begin{aligned} \frac{A}{\pi r^2} &= \frac{10\pi/3}{2\pi r} \\ A &= \frac{5\pi r}{3} = \frac{5\pi(4)}{3} = \frac{20\pi}{3} \text{ units}^2 \end{aligned}$$

Q7.c

Q7.c.i



Q7.c.ii

$$P(G) = \frac{2+2+3}{27} = \frac{7}{27}$$

Q7.c.iii

$$P(3 \text{ games}) = \frac{4+2+4+2}{27} = \frac{12}{27} = \frac{4}{9}$$

Question 8

Q8.a

Q8.a.i

Solving $f(x) = 0$ gives

$$\begin{aligned} x^4 - 8x^2 &= 0 \\ x^2(x^2 - 8) &= 0 \end{aligned}$$

Hence $x = 0$ or $x = \pm\sqrt{8} = \pm 2\sqrt{2}$. The corresponding coordinates are then found to be $(0,0), (-2\sqrt{2}, 0), (2\sqrt{2}, 0)$.

Q8.a.ii

The function is even if $f(-x) = f(x)$ for all x inthe domain.

Start, $f(-x) = (-x)^4 - 8(-x)^2 = x^4 - 8x^2 = f(x)$ (since a negative number raised to an even power is positive). Hence $f(x)$ is even.

Q8.a.iii

The first derivative is $f'(x) = 4x^3 - 16x = 4x(x^2-4)$. For stat points solve $f'(x) = 0$. So solve $4x(x^2-4) = 0$. This gives $x = 0, \pm 2$ and so the coordinates of the stationary points are; $(-2, -16), (0, 0), (2, -16)$.

$f''(x) = 12x^2 - 16$

$f''(0) = -16 < 0 \implies$ concave down so its a maximum. $f''(\pm 2) = 12 \times 2^2 - 16 = 32 > 0 \implies$ concave up so its a min. Concluding, $(-2, -16), (2, -16)$ are both minimum turning points, and $(0, 0)$ is a maximum turning point.

Q8.a.iv

The graph follows,

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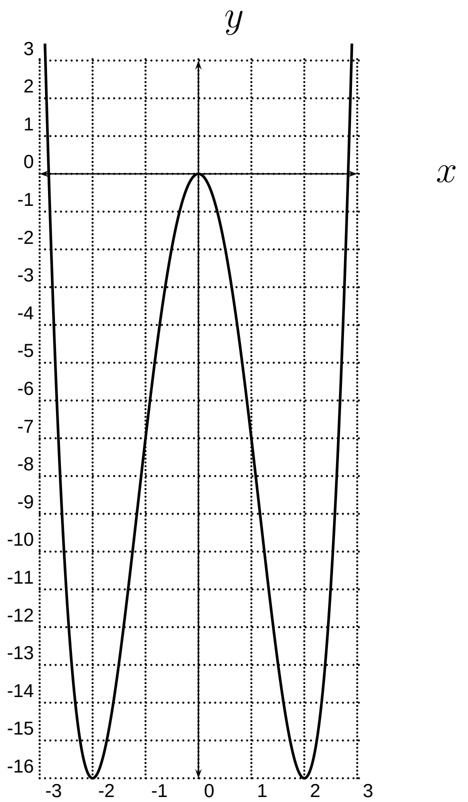

Q8.b

Q8.b.i

$AB = CD$ ($ABCD$ is a parallelogram)

$EB = AB$ ($ABCD$ is a square)

$\therefore CD = ED$ as required.

Q8.b.ii

$\angle DCB = \pi - \angle ABC$
(Cointerior angles in $\parallel$ lines $AB \parallel CD$ are supp.)
$\angle EBH = 2\pi - \pi/2 - \pi/2 - \angle ABC$
(Angles in a revolution, and given that $\angle ABE = \angle CBH = \pi/2$.)
$= \pi - \angle ABC$
$\angle DCB = \angle EB$
$AB = DC$
(Opposite sides of a parallelogram.)
$EB = AB$ ($ABEF$ is a square.)
$\therefore EB = DC$
$BH = BC$ ($BCGH$ is a square.)
$\Delta EBH \equiv \Delta DCB$ (SAS)

Hence $BD = EH$ (Matching sides in congruent triangles).

Question 9

Q9.a

Q9.a.i

$0.15^2 = 0.0225$.

Q9.a.ii

$0.85 \times 0.20 = 0.17$

Q9.b

Firstly 6% p.a. $= \frac{6}{12}$% per month $= 0.5$% per month $= 0.005$ per month (as a decimal)

Q9.b.i

$$\begin{aligned}
A_1 &= 100000 \times 1.005 - M \\
A_2 &= 100000 \times 1.005^2 - M \times 1.005 - M \\
A_3 &= 100000 \times 1.005^3 - M \times 1.005^2 - M \times 1.005 - M \\
&\vdots \\
A_n &= 100000 \times 1.005^n - M(1 + 1.005 + \cdots + 1.005^{n-1}) \\
A_n &= 100000 \times 1.005^n - M\frac{1.005^n - 1}{0.005}
\end{aligned}$$

Q9.b.ii

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We are given that $A_{144} = 0$ and so we solve the following, for M,

$$0 = 100000 \times 1.005^{144} - M\frac{1.005^144 - 1}{0.005}$$

$$M = \frac{100000 \times 1.005^{144} \times 0.005}{1.005^{144} - 1}$$

$$M = \$\mathbf{975.85}$$

Q9.c

$f''(x) = k(b^2 - x^2)$

Q9.c.i

$f'(x) = kb^2x - kx^3/3 + c$ and since $f'(-b) = -f'(b)$ then we have $kb^2(-b) - k(-b)^3/3 + c = -kb^3 + kb^3/3 - c \implies c = 0$.

$\therefore f'(x) = k(b^2x - x^3/3)$ as required.

Q9.c.ii

Integrating, $f(x) = k(b^2x^2/2 - x^4/12) + c$ and $f(b) = 0 \implies k(b^4/2 - b^4/12) + c = 0 \implies c = -5kb^4/4$.

Updating we have, $f(x) = k\left(\frac{b^2x^2}{3} - \frac{x^4}{12}\right) - \frac{5kb^4}{4}$. Hence $f(0) = -\frac{5kb^4}{4}$.

$\therefore$ the beam is $\frac{5kb^4}{4}$ units below the x-axis at $x = 0$.

Question 10

Q10.a

$$\begin{aligned}
y &= \log_e(x-2)\\
e^y &= x - 2\\
x &= e^y + 2\\
\text{Area} &= \int_0^{\ln 5} 7 - (e^y + 2)dy\\
&= \int_0^{\ln 5} 5 - e^y dy\\
&= [5y - e^y]_0^{\ln 5}\\
&= (5\ln 5 - e^{\ln 5}) - (5 \times 0 - e^0)\\
&= 5\ln 5 - 5 + 1\\
&= 5\ln 5 - 4
\end{aligned}$$

Q10.b

Q10.b.i

Let H and h be the perpendicular distances from O to MP, KJ respectively.
Then $\sin\alpha = h/x \implies h = x\sin\alpha$
$\sin\alpha = H/(l-x) \implies H = (l-x)\sin\alpha$.

$$\begin{aligned}
\text{Area} &= \tfrac{1}{2}.s.x.\sin\alpha + \tfrac{1}{2}\tfrac{(l-x)s}{x} \times (l-x)\sin\alpha\\
&= \tfrac{1}{2}s\sin\alpha\left(x + (l-x)^2/x\right)\\
&= \tfrac{s}{2}\sin\alpha\left(\tfrac{x^2+l^2-2xl+x^2}{x}\right)\\
&= \tfrac{s}{2}\sin\alpha\left(\tfrac{2x^2+l^2-2xl}{x}\right)\\
&= \tfrac{s}{2}\sin\alpha\left(2x + \tfrac{l^2}{x} - 2l\right)\\
&= s\sin\alpha\left(x - l + \tfrac{l^2}{2x}\right)
\end{aligned}$$

as required.

Q10.b.ii

$$\begin{aligned}
A &= s\sin\alpha\left(x - l + \tfrac{l^2}{2x}\right)\\
\frac{dA}{dx} &= s\sin\alpha\left(1 - \tfrac{l^2x^{-2}}{2}\right) = 0 \text{ for stat points}\\
1 &= l^2/(2x^2)\\
x^2 &= l^2/2\\
x &= l/\sqrt{2}
\end{aligned}$$

and this corresponds to a minimum value as $A'' = s\sin\alpha(2l^2x^{-3}/2) > 0$

Q10.b.iii

$MP = s(l-x)/x = s(l - l/\sqrt{2})/(l/\sqrt{2}) = s\sqrt{2}(1 - 1/\sqrt{2}) = s(\sqrt{2} - 1)$

The End

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