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# 2010 HSC exam - Mathematics Extension 2

# Question 8 Solution

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**The Basel Problem:** Prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}$.

**Note:** This was first proved by Euler. However this HSC exam does not follow Euler's method. It follows a method of Yoshio Matsuoka, Kagoshima University, Japan. This was published in 1961 in

Matsuoka, Y., *An Elementary Proof of the Formula* $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, The American Mathematical Monthly, **Vol. 68, No. 5 (May, 1961)**, pp. 485-487.

and can be downloaded at

http://www.angelfire.com/ab7/fourunit/matsuoka.pdf

**Proof:** Let $A_n = \int_0^{\pi/2} \cos^{2n} x\, dx$ & $B_n = \int_0^{\pi/2} x^2 \cos^{2n} x\, dx$ where $n$ is an integer, $n \geq 0$.

$$\begin{aligned}
\text{Then } \int_0^{\pi/2} \cos^{2n} x\, dx &= \int_0^{\pi/2} \cos^{2n-1} x \cos x\, dx \\
&= \int_0^{\pi/2} \cos^{2n-1} x \tfrac{d}{dx} \sin x\, dx \\
&= [\sin x \cos^{2n-1} x]_0^{\pi/2} - \int_0^{\pi/2} \sin x \tfrac{d}{dx} \cos^{2n-1} x\, dx \\
&= 0 - 0 - \int_0^{\pi/2} \sin x (2n-1) \cos^{2n-2} x (-\sin x)\, dx \\
&= (2n-1) \int_0^{\pi/2} \cos^{2n-2} x (1 - \cos^2 x)\, dx \\
&= (2n-1) \int_0^{\pi/2} \cos^{2n-2} x\, dx - (2n-1) \int_0^{\pi/2} \cos^{2n} x\, dx
\end{aligned}$$

$\therefore 2n \int_0^{\pi/2} \cos^{2n} x\, dx = (2n-1) \int_0^{\pi/2} \cos^{2n-2} x\, dx$ and

$n \int_0^{\pi/2} \cos^{2n} x \cos^{2n} x\, dx = \frac{2n-1}{2} \int_0^{\pi/2} \cos^{2(n-1)} x\, dx$ & $\therefore nA_n = \frac{2n-1}{2} A_{n-1}$ for $n \geq 1$.

$$\begin{aligned}
\text{Also, } A_n &= \int_0^{\pi/2} \cos^{2n} x \cdot \tfrac{dx}{dx}\, dx \\
&= [x \cos^{2n} x]_0^{\pi/2} - \int_0^{\pi/2} x \cdot 2n \cos^{2n-1} x (-\sin x)\, dx \\
&= 0 - 0 + 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x\, dx \\
&= 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x\, dx,\ \ n \geq 1
\end{aligned}$$

Hence, $A_n = 2n\int_0^{\pi/2} \sin x \cos^{2n-1} x \frac{d}{dx}(\frac{x^2}{2})\,dx$

$$= 2n([\tfrac{x^2}{2}\sin x\cos^{2n-1}x]_0^{\pi/2} - \int_0^{\pi/2}\tfrac{x^2}{2}\tfrac{d}{dx}(\sin x\cos^{2n-1}x)\,dx)$$
$$= 2n(0 - 0 - \int_0^{\pi/2}\tfrac{x^2}{2}((2n-1)\sin x\cos^{2n-2}x(-\sin x) + \cos^{2n-1}x\cos x)\,dx)$$
$$= n\int_0^{\pi/2} x^2((2n-1)\sin^2 x\cos^{2n-2}x - \cos^{2n}x)\,dx$$
$$= n\int_0^{\pi/2} x^2((2n-1)(1-\cos^2 x)\cos^{2n-2}x - \cos^{2n}x)\,dx$$
$$= n\int_0^{\pi/2} x^2((2n-1)\cos^{2n-2}x - (2n-1)\cos^{2n}x - \cos^{2n}x)\,dx$$
$$= n\int_0^{\pi/2} x^2((2n-1)\cos^{2n-2}x - 2n\cos^{2n}x)\,dx$$
$$= n(2n-1)\int_0^{\pi/2} x^2\cos^{2n-2}x\,dx - 2n^2\int_0^{\pi/2} x^2\cos^{2n}x\,dx$$
$$= n(2n-1)B_{n-1} - 2n^2 B_n$$

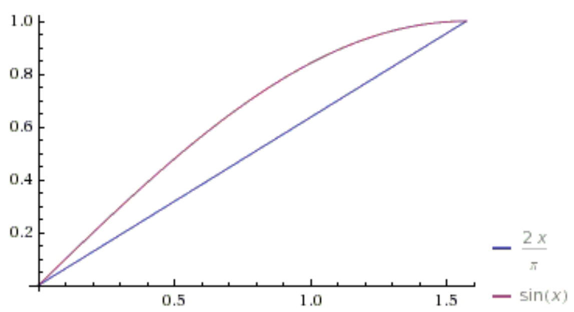
$\therefore \frac{A_n}{n^2} = \frac{2n-1}{n}B_{n-1} - 2B_n,\ n \geq 1$

$$\therefore \frac{1}{n^2} = \frac{2n-1}{nA_n}B_{n-1} - 2\frac{B_n}{A_n}$$
$$= \frac{2n-1}{n\cdot\frac{2n-1}{2n}A_{n-1}}B_{n-1} - 2\frac{B_n}{A_n}$$
$$= 2\frac{B_{n-1}}{A_{n-1}} - 2\frac{B_n}{A_n}$$
$$= 2(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}),\ n \geq 1$$

So $\sum_{k=1}^n \frac{1}{k^2} = 2(\frac{B_0}{A_0} - \frac{B_1}{A_1}) + 2(\frac{B_1}{A_1} - \frac{B_2}{A_2}) + \cdots + 2(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}) = 2\frac{B_0}{A_0} - 2\frac{B_n}{A_n}$.

But $\frac{2B_0}{A_0} = \frac{2\int_0^{\pi/2} x^2\,dx}{\int_0^{\pi/2} dx} = \frac{2[\frac{x^3}{3}]_0^{\pi/2}}{[x]_0^{\pi/2}} = \frac{2\cdot\frac{\pi^3}{24} - 0}{\frac{\pi}{2} - 0} = \frac{\pi^2}{6}$ and $\therefore\ \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$.

Behold:




$\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$ whereupon $\sin^2 x \geq \frac{4}{\pi^2}x^2$ and $1 - \cos^2 x \geq \frac{4}{\pi^2}x^2$ and so $\cos^2 x \leq 1 - \frac{4}{\pi^2}x^2$ & $\cos^{2n} x \leq (1 - \frac{4}{\pi^2}x^2)^n$. So $B_n \leq \int_0^{\pi/2} x^2(1 - \frac{4x^2}{\pi^2})^n\,dx$.

Noting that

$$\frac{\pi^2}{8(n+1)}\int_0^{\pi/2}(1 - \tfrac{4x^2}{\pi^2})^{n+1}\,dx$$
$$= \frac{\pi^2}{8(n+1)}\int_0^{\pi/2}(1 - \tfrac{4x^2}{\pi^2})^{n+1}\frac{dx}{dx}\,dx$$
$$= \frac{\pi^2}{8(n+1)}[x(1 - \tfrac{4x^2}{\pi^2})^{n+1}]_0^{\pi/2} - \frac{\pi^2}{8(n+1)}\int_0^{\pi/2} x\cdot(n+1)(1 - \tfrac{4x^2}{\pi^2})^n(\tfrac{-8x}{\pi^2})\,dx$$
$$= 0 - 0 + \int_0^{\pi/2} x^2(1 - \tfrac{4x^2}{\pi^2})^n\,dx$$

& $\therefore\ \int_0^{\pi/2} x^2(1-\frac{4x^2}{\pi^2})^n\,dx = \frac{\pi^2}{8(n+1)}\int_0^{\pi/2}(1-\frac{4x^2}{\pi^2})^{n+1}\,dx$

we now have that $B_n \le \frac{\pi^2}{8(n+1)}\int_0^{\pi/2}(1-\frac{4x^2}{\pi^2})^{n+1}\,dx$

Let $x = \frac{\pi}{2}\sin t \therefore dx = \frac{\pi}{2}\cos t\,dt$.

When $x = 0,\ t = 0$ & when $x = \frac{\pi}{2},\ t = \frac{\pi}{2}$.

$1 - \frac{4x^2}{\pi^2} = 1 - \frac{4}{\pi^2}\cdot\frac{\pi^2}{4}\sin^2 t = \cos^2 t.$

For $0 \le t \le \frac{\pi}{2},\ 0 \le \cos^3 t \le 1.$

$$\begin{aligned}\therefore B_n &\le \frac{\pi^2}{8(n+1)}\int_0^{\pi/2}\cos^{2n+2} t\cdot\frac{\pi}{2}\cos t\,dt\\ &= \frac{\pi^3}{16(n+1)}\int_0^{\pi/2}\cos^{2n+3} t\,dt\\ &= \frac{\pi^3}{16(n+1)}\int_0^{\pi/2}\cos^{2n} t\cdot\cos^3 t\,dt\\ &\le \frac{\pi^3}{16(n+1)}\int_0^{\pi/2}\cos^{2n} t\,dt\\ &= \frac{\pi^3}{16(n+1)}A_n\end{aligned}$$

$\therefore B_n \le \frac{\pi^3}{16(n+1)}\int_0^{\pi/2}\cos^{2n+3} t\,dt \le \frac{\pi^3}{16(n+1)}A_n.$

Noting $A_n > 0$ & $B_n > 0$, $0 < 2\frac{B_n}{A_n} \le 2\cdot\frac{\pi^3}{16(n+1)} = \frac{\pi^3}{8(n+1)}$

$\therefore -\frac{\pi^3}{8(n+1)} \le -2\frac{B_n}{A_n} < 0$

$\therefore \frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \le \frac{\pi^2}{6} - 2\frac{B_n}{A_n} = \sum_{k=1}^n \frac{1}{k^2} < \frac{\pi^2}{6}$

$\therefore \frac{\pi^2}{6} = \lim_{n\to\infty}(\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)}) \le \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k^2} \le \frac{\pi^2}{6}$

$\therefore \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6}$ $\square$