Scots Yr 12 Ext 1 Trial HSC 2011

SOLUTIONS

ANSWER BOOKLET

Name: ____________

Teacher: ____________

Question No. 1

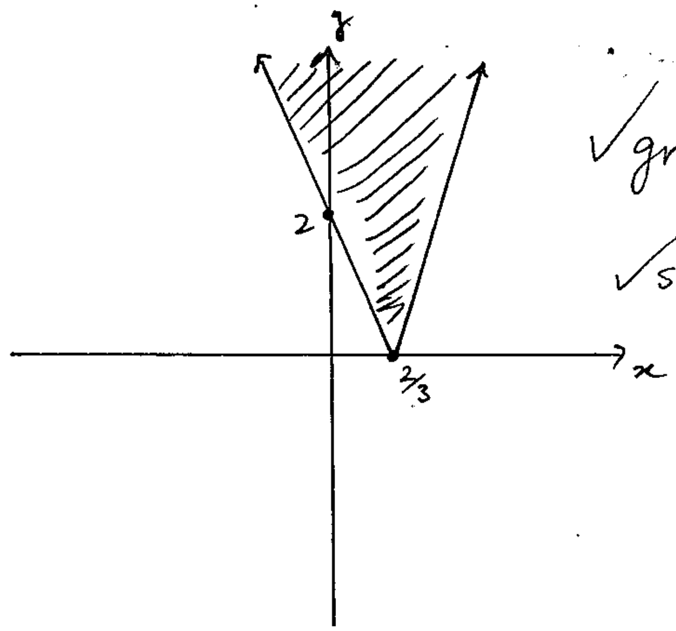
a) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C$ ✓ solution

b) $y \geq |3x-2|$,

test $(0,0)$

$0 \geq |0-2|$

$0 \geq 2$ Not true



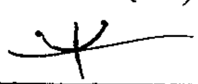
✓ graph

✓ shading

c) Note:

$y = \sin^{-1}(x)$

$y = \sin^{-1}(x^2)$



$\therefore$ $y = \sin^{-1}(x^2)$

domain : $-1 \leq x \leq 1$ ✓

Range : $0 \leq y \leq \frac{\pi}{2}$ ✓

ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 1

$u = 2x^5 - 1$

$\frac{du}{dx} = 10x^4$

$du = 10x^4\,dx$

d) $\int x^4(2x^5-1)^3\,dx$

$\frac{1}{10}\int 10x^4(2x^5-1)^3\,dx$ ✓ correct substitution

$\frac{1}{10}\int u^3\,du$

$\frac{1}{10}\left[\frac{u^4}{4} + c\right]$ ✓ integral

$\frac{u^4}{40} + c$

$\frac{(2x^5-1)^4}{40} + c$

✓ solution

Note: $+C$ not put down loss of 1 mark.

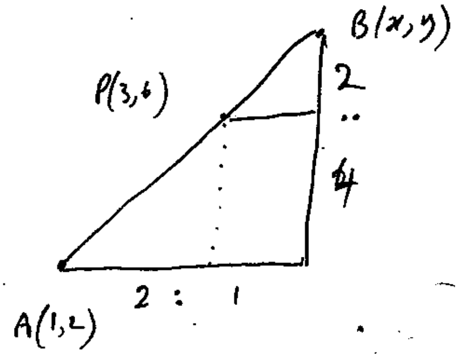
ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 1

e) $P(3,6)$



$\boxed{B(4,8)}$

✓ correct method

✓ solution

f) $y = 2x - 3$, $y = mx + 1$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan 30 = \left|\frac{2-m}{1+2m}\right|$$

✓ formula with values & two possible outcomes

$\therefore$

$$-\frac{1}{\sqrt{3}} = \frac{2-m}{1+2m} \quad \text{or} \quad \frac{1}{\sqrt{3}} = \frac{2-m}{1+2m}$$

$$m = \frac{-1-2\sqrt{3}}{2-\sqrt{3}} \qquad m = \frac{2\sqrt{3}-1}{2+\sqrt{3}}$$

$$m = -5\sqrt{3} - 8 \qquad m = 5\sqrt{3} - 8$$

✓ solution

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 2

$$\frac{d}{dx}\left(3\sin^{-1}(4x)\right)$$

let

$y = 3\sin^{-1}(4x)$ find $\frac{dy}{dx}$

let $u = 4x$ $\frac{du}{dx} = 4$

$y = 3\sin^{-1}(u)$

$$\frac{dy}{du} = \frac{3}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{12}{\sqrt{1-16x^2}}$$ ✓ ✓

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 2

b) $t=0,\ x=2,\ v=0$

$\ddot{x} = x+2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x+2$

$\frac{1}{2}v^2 = \frac{x^2}{2} + 2x + c$ ✓ using substitution & finding c.

when $x=2,\ v=0$

$0 = \frac{4}{2} + 4 + c$

$0 = 6 + c$

$-6 = c$

$\therefore$ ✓ attempting to solve for V.

$\frac{1}{2}v^2 = \frac{x^2}{2} + 2x - 6$

$\therefore$ at $x=4$

$\frac{1}{2}v^2 = \frac{16}{2} + 8 - 6$

$\frac{1}{2}v^2 = 10$

$v = \pm\sqrt{20}$ $\therefore$ $V = \pm\sqrt{20}$ ✓ solution

Speed is $\sqrt{20}$ when $x=4$.

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 2

c) i) $e^{2x}(\sin x + 2\cos x)$

$u \times v$

✓ attempts product rule

$\frac{d}{dx} = 2e^{2x}(\sin x + 2\cos x) + e^{2x}(\cos x - 2\sin x)$

$= 4e^{2x}\cos x + e^{2x}\cos x$

$= 5e^{2x}\cos x$ ✓ solution

$u = e^{2x}$ $u' = 2e^{2x}$

$v = \sin x + 2\cos x$ $v' = \cos x - 2\sin x$

ii) $\int e^{2x}\cos x \, dx$

$\frac{1}{5}\int 5e^{2x}\cos x \, dx$

$\frac{1}{5}\left[e^{2x}(\sin x + 2\cos x)\right] + c$

$\frac{1}{5}e^{2x}(\sin x + 2\cos x) + c$ ✓ solution

ANSWER BOOKLET

Name: ________________

Teacher: ________________

Question No. 2

d) $x = 2t$, $y = 2t^2$

$\frac{x}{2} = t$

$\therefore$

$y = 2\left(\frac{x}{2}\right)^2$

$= \frac{2x^2}{4}$

$y = \frac{x^2}{2}$

✓ solving for t

✓ equating

e) $\sum_{n=6}^{8} (3n-1) = [3\times6-1] + [3\times7-1] + [3\times8-1]$

$= 17 + 20 + 23$

$= 60$

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 2

f)

$$\lim_{x \to 0} \frac{\sin 2x}{x}$$

$$\lim_{x \to 0} \frac{2\sin 2x}{2x}$$

$$\lim_{x \to 0} 2 \times \frac{\sin 2x}{2x}$$

$$\lim_{x \to 0} 2 \times \lim_{x \to 0} \frac{\sin 2x}{2x}$$

$$2 \times 1$$

$$2$$

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 3

a) i) $e^{x} - \sin x - 3 = ?$

at $x = 1$

$e^{1} - \sin(1) - 3 = -1.12$

at $x = 2$

$e^{2} - \sin(2) - 3 = 3.48$

$\therefore$ change of sign there exists a root between $x = 1$ & $x = 2$.

ii) Newtons' $\boxed{x = a - \frac{f(a)}{f'(a)}}$

$f(x) = e^{x} - \sin x - 3$

$f'(x) = e^{x} - \cos x$

$f(1.5) = 0.484$

$f'(1.5) = 4.41095$

at $a = 1.5$

$x = 1.5 - \frac{0.484}{4.41095}$

$x = 1.5 - 0.1098$

$x = 1.39$

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 3

b i) $x = 3\cos(2t + \frac{\pi}{4})$ where period is π

$\therefore \quad P = \frac{2\pi}{n}$

$n = 2$

$\dot{x} = -6\sin(2t + \frac{\pi}{4})$

$\ddot{x} = -12\cos(2t + \frac{\pi}{4})$

$\ddot{x} = -4 \times 3\cos(2t + \frac{\pi}{4})$

$\ddot{x} = -4x$

$\therefore$ this is in the form

$\ddot{x} = -n^2 x$

$\therefore$ particle is undergoing SHM.

b ii) amplitude is 3.

b iii) $\dot{x} = -6\sin(2t + \frac{\pi}{4})$

$-6 = -6\sin(2t + \frac{\pi}{4})$

$+1 = \sin(2t + \frac{\pi}{4})$

$\therefore$

$2t + \frac{\pi}{4} = \frac{\pi}{2}$

$t = \frac{\pi}{8}$

$\therefore$ at time $\frac{\pi}{8}$ particle first reaches its max speed.

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 3

6 i) $\sin 3\theta = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$

$= \sin\theta\left[1-2\sin^2\theta\right] + \cos\theta\left[2\sin\theta\cos\theta\right]$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\left(1-\sin^2\theta\right)$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^2\theta$.

$= 3\sin\theta - 4\sin^3\theta$

ii) $\sin 3\theta = \sin\theta$

$3\sin\theta - 4\sin^3\theta = \sin\theta$

$2\sin\theta = 4\sin^3\theta$, $\sin\theta = 0$

$\frac{2}{4} = \sin^2\theta$

$\pm\frac{1}{\sqrt{2}} = \sin\theta$

$\therefore$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4},$

General solutions $\theta = \left(\frac{\pi}{4} \pm n\pi\right)$ or $\left(\frac{3\pi}{4} \pm n\pi\right)$ where $(n = 0, 1, 2, \ldots)$

or $n\pi$

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 4

$P(x) = ax^3 + bx^2 + cx + d$

$\alpha+\beta+\gamma = \frac{-b}{a}$

$\alpha\beta+\gamma\alpha+\beta\gamma = \frac{c}{a}$

$\alpha\beta\gamma = \frac{-d}{a}$

a) i) $\alpha+\beta+\gamma = \frac{5}{2}$

ii) $\alpha\beta\gamma = \frac{-40}{2} = -20$

iii) let $\alpha = -\beta$

$\therefore$

$-\alpha^2\gamma = -20 \quad \cdots ①$

$\gamma = \frac{5}{2} \quad \cdots ②$

$-\alpha^2\left(\frac{5}{2}\right) = -20$

$5\alpha^2 = 40$

$\alpha^2 = 8$

$\alpha = \pm\sqrt{8}$

$\therefore$

$\alpha = \sqrt{8}\ ,\ \beta = -\sqrt{8}\ ,\ \gamma = \frac{5}{2}$

$\alpha\beta + \gamma\alpha + \beta\gamma = \frac{k}{2}$

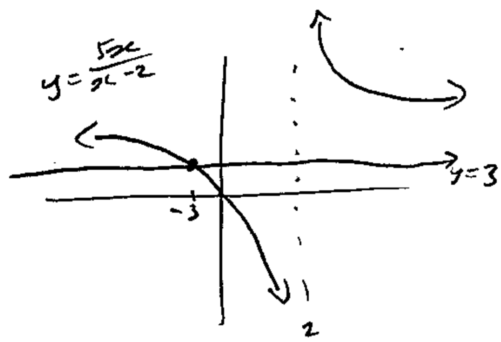
$-8 + \sqrt{8}\left(\frac{5}{2}\right) - \sqrt{8}\left(\frac{5}{2}\right) = \frac{k}{2}$

$-8 = \frac{k}{2}$

$-16 = k$

Question No. 4

b) $\dfrac{5x}{x-2} \leq 3 \qquad x \neq 2$



$5x(x-2) \leq 3(x-2)^2$

$5x^2 - 10x \leq 3(x^2-4x+4)$

$5x^2 - 10x \leq 3x^2 - 12x + 12$

$2x^2 + 2x - 12 \leq 0$

$2(x^2+x-6) \leq 0$

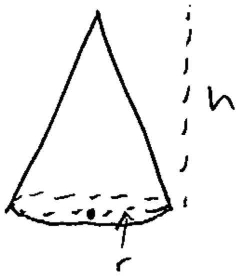
$2(x+3)(x-2) \leq 0$

$\therefore$

$-3 \leq x < 2$

Question No. 6

b) i



$2r = h$

$r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{12} h^3$

$\frac{dV}{dh} = \frac{3\pi}{12} h^2 = \frac{\pi}{4} h^2$

ii) $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$\frac{dV}{dt} = 7$

$\frac{dh}{dV} = \frac{4}{\pi h^2}$

$= 7 \times \frac{4}{\pi h^2}$

$\frac{dh}{dt} = \frac{28}{\pi h^2}$

when $h = 2$

$\frac{dh}{dt} = \frac{28}{\pi \times 4}$

$= \frac{7}{\pi}$

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 5

$\cos 2A = 1 - 2\sin^2 A$

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

a) $$\int_0^{\frac{\pi}{4}} \sin^2(4x)\,dx$$

$$\int_0^{\frac{\pi}{4}} \frac{1}{2} - \frac{1}{2}\cos(8x)\,dx$$

$$= \left[\frac{x}{2} - \frac{1}{16}\sin(8x)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{8} - \frac{1}{16}\sin(2\pi)\right] - \left[0 - 0\right]$$

$$= \left[\frac{\pi}{8} - 0\right] - 0$$

$$= \frac{\pi}{8}$$

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

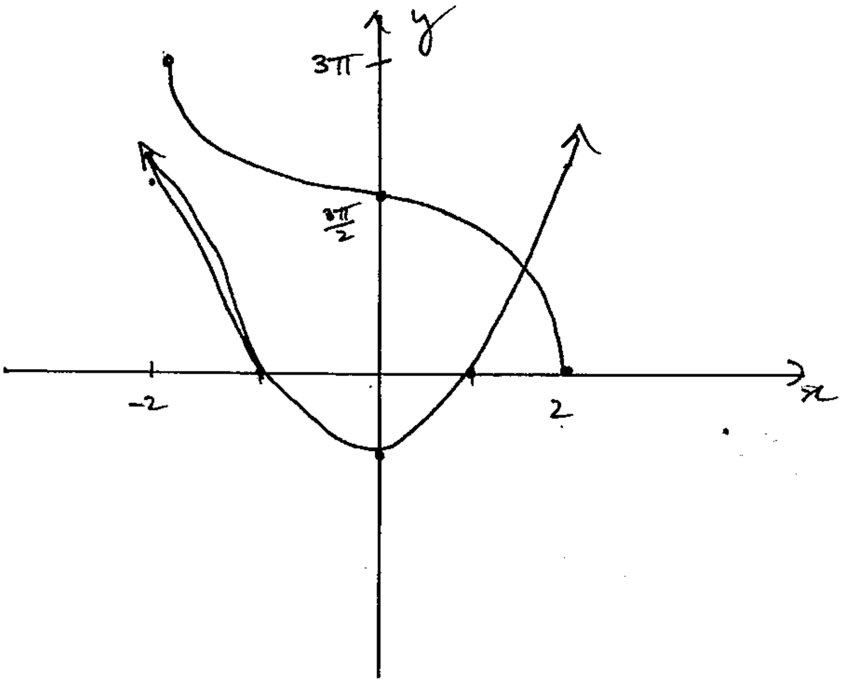
Question No. 5

b)

$$y = 3\cos^{-1}\left(\frac{x}{2}\right)$$

$$\frac{y}{3} = \cos^{-1}\left(\frac{x}{2}\right)$$

$$x = 2\cos\left(\frac{1}{3}y\right)$$



ii) $y = 2x^2 - 2$

at

x	-2	-1	0	1	2
y	6	0	-2	0	6

iii) on domain

$-2 \leq x \leq 2$

only 1 solution

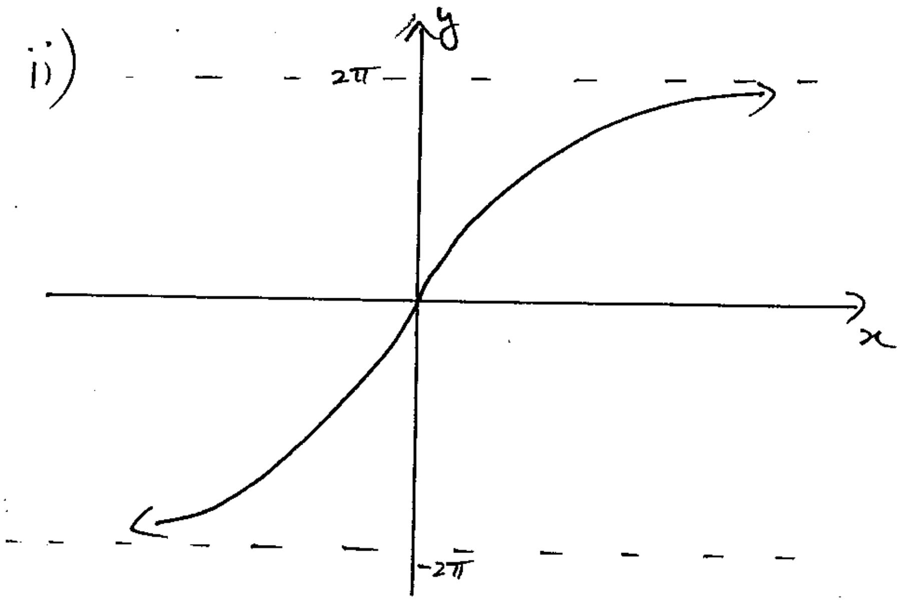
Question No. 5

c) i) range of $y=f(x)$

$-4\times\frac{\pi}{2} \le y \le 4\times\frac{\pi}{2}$

$\boxed{-2\pi < y < 2\pi}$

ii)



iii) $y = 4\tan^{-1}(x)$

$\frac{dy}{dx} = \frac{4}{1+x^2}$

at $x=\sqrt{3}$

$\frac{dy}{dx} = \frac{4}{4} = 1$

at $x=\sqrt{3}$

$y = 4\tan^{-1}(\sqrt{3})$

$= \frac{4\pi}{3}$

Tangent

$y - \frac{4\pi}{3} = 1(x-\sqrt{3})$

$3x - 3y - 3\sqrt{3} + 4\pi = 0$

ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 6

a) i $\frac{dE}{dt} = k(E-c)$

$\frac{dt}{dE} = \frac{1}{k} \times \frac{1}{(E-c)}$

$t = \frac{1}{k}\int \frac{1}{E-c} dE$

$t = \frac{1}{k}\ln(E-c) + D$

$kt - D = \ln(E-c)$

$e^{kt-D} = E-c$

$\frac{e^{kt}}{e^{D}} = E-c$ $\left(\text{let } \frac{1}{e^{D}} = A\right)$

$Ae^{kt} = E-c$

$\therefore$

$E = Ae^{kt} + c$

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Name: ______________________

Teacher: ______________________

Question No. 6

a) ii) $E = Ae^{kt} + c$

at $t=0$ $E=0$ $c=8\cdot3$

$0 = Ae^{0} + 8\cdot3$

$A = -8\cdot3$

$\therefore$

$E = -8\cdot3e^{kt} + 8\cdot3$

at $t=3$ $E=2\cdot9$ $c=8\cdot3$

$2\cdot9 = -8\cdot3e^{3k} + 8\cdot3$

$\frac{-5\cdot4}{-8\cdot3} = e^{3k}$

$\frac{54}{83} = e^{3k}$

$k = \frac{1}{3}\ln\left(\frac{54}{83}\right)$

$\therefore$

$E = -8\cdot3e^{kt} + 8\cdot3$

at $t=5$

$E = -8\cdot3e^{5k} + 8\cdot3$

$= 4\cdot245$

$= 4\cdot2$ (2 sig fig)

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 6

b)

$$V = \pi \int_3^a \left(3 + \frac{1}{2x-5}\right)^2 dx$$

$$= \pi \int_3^a 9 + \frac{6}{2x-5} + (2x-5)^{-2}\, dx$$

$$= \pi \left[9x + 3\ln(2x-5) + -\tfrac{1}{2}(2x-5)^{-1} \right]_3^a$$

$$= \pi \left[9x + 3\ln(2x-5) - \frac{1}{4x-10} \right]_3^a$$

$$= \pi \left[\left(9a + 3\ln(2a-5) - \frac{1}{4a-10}\right) - \left(27 + 0 - \frac{1}{2}\right) \right]$$

$$= \pi \left[9a - 27 + \frac{1}{2} - \frac{1}{4a-10} + 3\ln(2a-5) \right]$$

$$\therefore \quad 2a - 5 = 3$$
$$a = 4$$

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 6

b) $\therefore$ if $a = 4$

$9 \times 4 - 27 + \frac{1}{2} - \frac{1}{16 - 10} =$

$9\frac{1}{2} - \frac{1}{6} =$

$\frac{28}{3}$ = as required.

ANSWER BOOKLET

Name: ______________________

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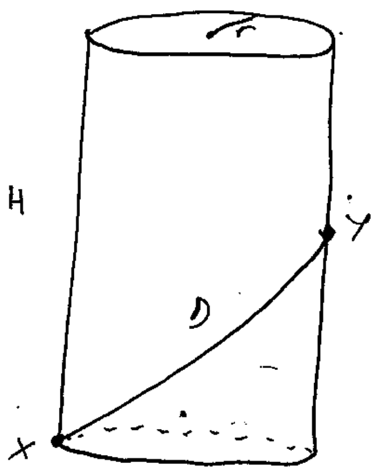
Question No. 6

d) i)

$$D^2 = \frac{H^2}{4} + 4r^2$$

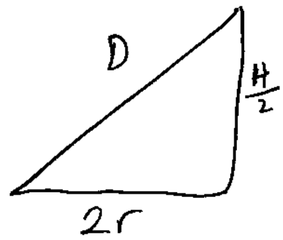
$$r^2 = \frac{D^2}{4} - \frac{H^2}{16}$$



$$V = \pi r^2 H$$

$$V = \pi\left(\frac{D^2}{4} - \frac{H^2}{16}\right)H$$

$$V = \frac{\pi H}{16}\left(4D^2 - H^2\right)$$



ii)

$$V = -\frac{\pi}{16}H^3 + \frac{\pi D^2}{4}H$$

$$\frac{dV}{dH} = -\frac{3\pi}{16}H^2 + \frac{\pi D^2}{4}$$

at $\frac{dV}{dH} = 0$ is max or min

cubic is of form

$y = -ax^3 + bx$

$\therefore$ take +ve solution

$$0 = -\frac{3\pi}{16}H^2 + \frac{\pi D^2}{4}$$

$$H^2 = \frac{4D^2}{3}$$

$$H = \pm\sqrt{\frac{4D^2}{3}}$$

only +ve solution

$$H = \frac{2D}{\sqrt{3}}$$

ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 7

a) Test for $n=1$

$$\left(1-\frac{1}{2^2}\right) = \frac{1+2}{2+2}$$

$$\frac{3}{4} = \frac{3}{4}$$

$\therefore$ True for $n=1$

We assume it is true for $n=k$ $(k \in \mathbb{N})$

i.e.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$$

Now prove it is true for $n=k+1$

$$\underbrace{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)}_{\text{from assumption } \frac{k+2}{2k+2}}\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+3}{2k+4}$$

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 7

a) continued

$$\left(\frac{k+2}{2k+2}\right)\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+3}{2k+4}$$

✓ progress

$$\frac{k+2}{2k+2} - \frac{k+2}{(2k+2)(k+2)^2}$$

$$\frac{(k+2)(k+2)^2-(k+2)}{(2k+2)(k+2)^2}$$

$$\frac{[k+2][(k+2)^2-1]}{2(k+2)^2(k+1)}$$

$$\frac{(k+2)^2-1}{2(k+2)(k+1)}$$

$$\frac{k^2+4k+4-1}{(2k+4)(k+1)}$$

$$\frac{(k+1)(k+3)}{(k+1)(2k+4)}$$

$$\frac{k+3}{2k+4} = RHS$$

∴ true for $n=k+1$

∴ Because it is true for $n=1$ it is true for $n=2$ and by mathematical induction is true for $n=3, 4$ etc

✓ solution.

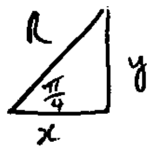
ANSWER BOOKLET

Name:

Teacher:

Question No. 7

b i)



iscosoles

$\therefore x = y$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} = \frac{y}{R}$

$\frac{R}{\sqrt{2}} = y$

$\therefore$

$x = y = \frac{R}{\sqrt{2}}$

ii)

$x = y$

$10t\cos\theta = -5t^2 + 10t\sin\theta$

$\therefore$

$5t^2 - 10t\sin\theta + 10t\cos\theta = 0$ ✓ setup

$5t\left(t - 2\sin\theta + 2\cos\theta\right) = 0$

$\therefore$ $t = 0$

or

$t - 2\sin\theta + 2\cos\theta = 0$

$\therefore$

$t = 2\sin\theta - 2\cos\theta$ ✓ solve for t

$\therefore$

$x = 10t\cos\theta$

when $t = 2\sin\theta - 2\cos\theta$

$x = 10(2\sin\theta - 2\cos\theta)\cos\theta$

but

$x = \frac{R}{\sqrt{2}}$

✓ substitute in for x

$\frac{R}{\sqrt{2}} = 20(\sin\theta\cos\theta - \cos^2\theta)$

$R = 20\sqrt{2}(\cos\theta\sin\theta - \cos^2\theta)$

As required

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Name: ____________

Teacher: ____________

Question No. 7

iii)

$R = 20\sqrt{2}\left(\underbrace{\cos\theta\sin\theta}_{\text{Product Rule}} - \underbrace{\cos^2\theta}_{\text{Product Rule}}\right)$

$\frac{dR}{d\theta} =$

$\frac{dR}{d\theta} = 0$ for max.

$0 = 20\sqrt{2}(\cos 2\theta + \sin 2\theta)$ ✓ $\frac{dR}{d\theta}$

$\therefore$

$\cos 2\theta + \sin 2\theta = 0$

$\sin 2\theta = -\cos 2\theta$

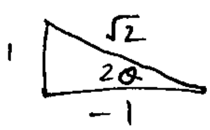
$\tan 2\theta = -1$

S	A
T	C

2nd or 4th but θ must be in domain $\frac{\pi}{4} < \theta \leq \frac{\pi}{2}$

$\frac{\pi}{2} < 2\theta < \pi$

$\therefore$ 2nd



$\therefore \cos 2\theta = \frac{-1}{\sqrt{2}}$ ✓ solution

$\sin 2\theta = \frac{1}{\sqrt{2}}$

$R = 20\sqrt{2}(\cos\theta\sin\theta - \cos^2\theta)$

Note:

$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2\sqrt{2}}$

$\cos^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta = \frac{1}{2} - \frac{1}{2\sqrt{2}}$

$R = 20\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)\right)$

$= 20\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$

$= 20\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$

$= 20 - 10\sqrt{2}$ ✓ sub in value

Maximum value of R is $(20 - 10\sqrt{2})$ ✓ exact value.